A guide through the theory of symmetric spaces

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Chapter I

Introduction

I.1 Historical Introduction

The theory of symmetric spaces was initiated by E. Cartan in 1926. While he was studying Riemannian locally symmetric spaces, he discovered, via the paper by H. Weyl [Wey26], that the problem he was studying was equivalent to the one he had studied some twelve years earlier, namely the classification of real forms of complex semisimple Lie algebras.

The original definition of symmetric space belongs to the realm of Riemannian geometry, in that a Riemannian symmetric space was originally defined as a **Riemannian manifold whose curvature tensor is invariant under parallel translation**. While the Riemannian geometrical acception has not faded, Cartan discovered that symmetric spaces are as related to Riemannian geometry as they are to Lie groups.

There are at least three good reasons to study symmetric spaces:

- They connect seemingly different fields of mathematics, and hence each one of the fields can enhance the knowledge about the other. As Cartan put it: "The theory of groups and geometry, leaning on one another, allow one to take up and solve a great variety of problems", [Car26].
- Many well known examples are indeed symmetric spaces.
- They are beautiful!

Examples. (1) The **Euclidean n-space** $\mathbb{E} := (\mathbb{R}^n, g_{Eucl})$ is a symmetric space. Its sectional curvature vanishes everywhere. Its isometry group is $\mathrm{O}(n) \ltimes \mathbb{R}^n$.

(2) The **unit sphere** S^n in $\mathbb{R}^{n+1}$ equipped with the Riemannian metric induced by $\mathbb{R}^{n+1}$ is a symmetric space whose sectional curvature is everywhere equal to one. Its isometry group is $\mathrm{O}(n, \mathbb{R})$.

(3) Let $q\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be the quadratic form associated to the symmetric bilinear form of signature $(n,1)$

$$\langle x,y\rangle := x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}\,.$$

Then

$$\mathcal{H}^n_{\mathbb{R}} := \{x \in \mathbb{R}^{n+1} :\, q(x) = \langle x,x\rangle = -1 \text{ and } x_{n+1} > 0\}$$

is the **(real)**[1] **hyperbolic n-space**. To give it a metric, write for every $x \in \mathcal{H}^n_{\mathbb{R}}$

$$\mathbb{R}^{n+1} = \mathbb{R}x \oplus (\mathbb{R}x)^{\perp} \quad \text{where} \quad (\mathbb{R}x)^{\perp} := \left\{y \in \mathbb{R}^{n+1} : \langle x,y\rangle = 0\right\}.$$

Since $\langle x,x\rangle = -1$, the restriction $\langle\cdot,\cdot\rangle|_{(\mathbb{R}x)^{\perp}}$ is positive definite and hence defines a Riemannian metric on $\mathcal{H}^n_{\mathbb{R}}$. $\mathcal{H}^n_{\mathbb{R}}$ is a symmetric space whose sectional curvature is identically equal to -1. Its isometry group is $\mathrm{O}(n,1)_+$, where

$$\mathrm{O}(n,1) := \{g \in \mathrm{GL}(n+1,\mathbb{R}) :\, q(gx) = q(x) \text{ for every } x \in \mathbb{R}^{n+1}\}$$

and

$$\mathrm{O}(n,1)_+ := \{g \in \mathrm{O}(n,1) :\, g\mathcal{H}^n_{\mathbb{R}} = \mathcal{H}^n_{\mathbb{R}}\}\,.$$

In each of the above cases it is easy to see that the isometry group acts transitively on the symmetric space.

I.2 Overview

I.2.1 Riemannian Geometrical Characterization of Symmetric Spaces

[1] We could define also the complex and quaternionic hyperbolic space. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Recall that the quaternions $\mathbb{H}$ is a four dimensional algebra over $\mathbb{R}$ with basis $\{1,i,j,k\}$, where 1 is central, $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$. Endow the space $\mathbb{K}^{n+1}$ with the $\mathbb{K}$-Hermitian form q defined by

$$q(x,y) := \overline{x_1}y_1 + \cdots + \overline{x_n}y_n - \overline{x_{n+1}}y_{n+1}\,,$$

(where conjugation is of course trivial in $\mathbb{R}$). If $\mathbb{PK}^n$ is the projective space $\mathbb{PK}^n = (\mathbb{K}^{n+1}\setminus\{0\})/\mathbb{K}^*$, the set

$$\mathcal{H}^n_{\mathbb{K}} := \{x \in \mathbb{PK}^n :\, q(x,x) < 0\}\,.$$

is called **real**, **complex** or **quaternionic hyperbolic n-space** $\mathcal{H}^n_{\mathbb{K}}$, according to whether $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Its dimension is, accordingly, n, $2n$ or $4n$.

Convention. A manifold is always assumed to be connected, second countable, paracompact, Hausdorff and finite dimensional. The only exception are Lie groups, that are allowed to have several components.

If M is a Riemannian manifold and $p \in M$, a **geodesic symmetry** at p is a map defined in a neighborhood of p that fixes p and reverses any local geodesic through p.

Remark. A geodesic symmetry need not be an isometry and need not be defined on the whole of M.

Definition: Riemannian Symmetric Space

The Riemannian manifold M is **Riemannian locally symmetric** if for every $p \in M$, there exists a geodesic symmetry s_p that additionally is an isometry on its domain.
A Riemannian manifold is a **Riemannian glolbally symmetric space** if it is locally Riemannian symmetric and in addition for every $p \in M$ the geodesic symmetry s_p is defined on the whole of M.

Example. (1) As an exercise define the geodesic symmetry in the case of S^n and of Euclidean n-space.

(2) Let $\mathcal{H}^n_{\mathbb{K}}$ be hyperbolic n-space. We can identify[2] the tangent space $T_x\mathcal{H}^n_{\mathbb{K}}$ at the point $x \in \mathcal{H}^n_{\mathbb{K}}$ with $x^\perp := \{y \in \mathbb{K}^{n+1} : q(x,y) = 0\}$. The Hermitian form q has signature $(n,1)$ and $\mathbb{K}^{n+1} = (\mathbb{K}x) \oplus (\mathbb{K}x)^\perp$, so that the restriction of q to $x^\perp$ is positive definite. It follows that $\operatorname{Re} q(u,v)$ descends to an inner product on $T_x\mathcal{H}^n_{\mathbb{K}}$ that turns $\mathcal{H}^n_{\mathbb{K}}$ into a Riemannian manifold.

If for example $\mathbb{K} = \mathbb{R}$, then geodesics in this model are the intersection of the hyperboloid with planes through the origin. The geodesic symmetry is defined at x by

$$s_x(y) := -2xq(x,y) - y\,.$$

In fact, we will show that a geodesic symmetry is characterized by $s_x \in O(q,\mathbb{K})$, $(s_x)^2 = Id$, $s_x(x) = x$ and s_x preserves the Riemannian metric: namely, if $z \in \mathcal{H}^n_{\mathbb{K}}$, then $\mathrm{d}_z s_x \colon T_z\mathcal{H}^n_{\mathbb{K}} \to T_{s_x(z)}\mathcal{H}^n_{\mathbb{K}}$ has the property that

$$q(\mathrm{d}_z s_x(v), \mathrm{d}_z s_x(v)) = q(s_x(v), s_x(v)) = q(v,v)\,,$$

[2]Consider the map $F(x) := q(x,x) + 1$. If $x \in F^{-1}(0)$, then $\ker \mathrm{d}_x F = T_x(F^{-1}(0))$, and $(\mathrm{d}_x F)(y) = \frac{d}{dt}|_{t=0} F(x+ty) = 2q(x,y)$.

where we have used that the differential of a linear map is the linear map itself. If follows also that, if v is a tangent vector at x, then

$$s_x(v) = 2vq(x,v) - v = -v\,.$$

We will see that if M is Riemannian globally symmetric, then it is complete and the connected component of its isometry group is small enough to be finite dimensional, but large enough to act transitively. The stabilizer of a point is going to be a compact subgroup of $\mathrm{Iso}(M)^\circ$.

We next list a few more (and less well known) examples:

Example. (1) A compact semisimple Lie group can be turned into a Riemannian symmetric space.

(2) Any compact orientable Riemann surface of genus $g \geq 2$ is locally Riemannian symmetric but not Riemannian symmetric. They are all quotients $\mathcal{H}^2_{\mathbb{R}}/\Gamma$, where $\Gamma < \mathrm{Iso}(\mathcal{H}^2_{\mathbb{R}})^\circ$ is a discrete cocompact subgroup (isomorphic to the fundamental group of the surface).

(3) Quotients of 2-dimensional real hyperbolic space by $\mathrm{SL}(2,\mathbb{Z})$ or by any finite index subgroup are locally Riemannian symmetric with finite volume but not compact).

(4) Borel showed that any Riemannian symmetric space, whose isometry group is semisimple, admits a quotient that is of finite volume and compact (using number theoretical arguments).

In fact, developing the theory leads to the first fact that any symmetric space is of the form $\mathbb{R}^m \times G/K$, where $\mathbb{R}^m$ is a Euclidean space and G is a semisimple Lie group that has an involutive automorphism σ whose fixed point is essentially K (in fact, $(G^\sigma)^\circ \leq K \leq G^\sigma$).

It is clear from the above examples that the theory of Riemannian locally symmetric spaces is part of the realm of discrete subgroups of semisimple Lie groups. We will soon leave aside the Riemannian locally symmetric spaces and concentrate on the Riemannian globally symmetric ones.

I.2.2 Algebraic Characterization of Symmetric Spaces

A symmetric space can be characterized from a purely algebraic point of view as follows. Take a connected Lie group and $\sigma\colon G \to G$ an involutive automorphism $\sigma^2 = Id$. A **symmetric space** for G is a homogeneous space G/H such that $H < G^\sigma$ is an open subgroup (hence union of connected components). If the group G^σ of σ-fixed points is compact, then G^σ can be equipped with a Riemannian metric by considering any G^σ-invariant inner product on the tangent space at eG^σ (which is possible since G^σ is compact) and smearing it around using the G-action. If $(G^\sigma)^\circ \leq K \leq G^\sigma$, then G/K is a Riemannian symmetric space.

Remark. Differentiation of σ gives a decomposition of $\mathfrak{g}$ into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ is the eigenspace with eigenvalue $+1$ and $\mathfrak{m}$ is the eigenspace with eigenvalue -1. Then $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. These three conditions indeed are equivalent in turn to the existence of an involutive automorphism of G with $\mathfrak{h}$ as a $+1$ eigenspace and $\mathfrak{m}$ as a -1 eigenspace.

I.2.3 Equivalence between the two Characterizations

If M is Riemannian symmetric, then $M \cong G/K$, where $G = \mathrm{Iso}(M)^\circ$ and $K = \mathrm{Stab}_G(p)$, where $p \in M$ is any point. Then K is compact and $\sigma \colon G \to G$, defined by $\sigma(g) = s_p g s_p$ is an involutive automorphism of G such that $(G^\sigma)^\circ \subset K \subset G^\sigma$ (and is hence open).

To see the converse, that is that $M = G/K$ is Riemannian symmetric, we need to define $s_p \colon M \to M$, where $p = hK \in M$. For $l \in G$ we set $s_p(lK) = h\sigma(h^{-1}l)K$, where σ is the involution of G fixing K. One can then see that $s_p(p) = p$, $s_p \in \mathrm{Iso}(M)$ and $\mathrm{d}_p s_p \colon T_pM \to T_pM$ is just $\mathrm{d}_p s_p = -Id$.

I.2.4 Decomposition, Classification and More to Follow

In 1926 Cartan classified all simply connected Riemannian symmetric spaces. Using the de Rham decomposition, one can see that any simply connected Riemannian symmetric space can be written as a product of $M_0 \times M_+ \times M_-$, where

- M_0 has zero curvature and is hence isometric to $\mathbb{R}^n$;
- M_+ has non-negative sectional curvature;
- M_- has non-positive sectional curvature.

The simply connected symmetric spaces of non-negative curvature are those of **compact type**, while the M_- are of **non-compact type**. Both have semisimple isometry group. The compact and non-compact symmetric spaces are moreover dual one of the other (resembling the analogy between spherical geometry and hyperbolic geometry, that can be, in fact, explained by this duality).

An important invariant of a symmetric space is its **rank**. This can be explained from a Riemannian geometrical point of view as the maximal dimension of any totally geodesic subspace of M (that is the maximal dimension of a subspace of the tangent space to any point in which the curvature is zero). From the Lie theoretical point of view the rank is given in terms of the dimension of a Cartan subalgebra, that is a maximal Abelian subalgebra that is diagonalizable.

If the rank is one, the maximal flats are geodesics. Thus the curvature is either negative or positive, and we have as examples the hyperbolic spaces defined before (in negative curvature) and the sphere (in positive curvature).

Here we will focus mostly on symmetric spaces of non-compact type. In this case $K < G$ is a maximal compact subgroup (and all maximal compact are conjugate). We will also see various decompositions such as the Cartan and the Iwasawa decomposition. Finally we will study the geometry at infinity of a symmetric space.

I.2.5 (Maximal) Prerequisites in Riemannian Geometry

- Parallel transport, geodesic and the exponential map;
- Isometries of a Riemannian manifold as a metric space;
- de Rham decomposition;
- Levi-Civita connection;
- Curvatures (Riemann curvature tensor, sectional curvature);
- Jacobi fields.

I.2.6 Textbooks

(1) A. Borel, [Bor98]

(2) M. Bridson and A. Haefliger, [BH99]

(3) M. do Carmo, [dC92]

(4) P. Eberlein, [Ebe96]

(5) S. Helgason, [Hel01]

(6) S. Kobayashi and K. Nomizu, [KN96]

Chapter II

Generalities on Riemannian Globally Symmetric Spaces

II.1 Isometries and the Isometry Group

A *Riemannian metric* g on a smooth manifold M is a map that associates to every $x \in M$ a scalar product on T_xM such that for every coordinate chart $\varphi\colon U \to \mathbb{R}^n$ the function

$$\begin{aligned} U &\longrightarrow \mathbb{R} \\ x &\mapsto g_x\left((d_x\varphi)^{-1}(e_i), (d_x\varphi)^{-1}(e_j)\right) \qquad 1 \le i, j \le n \end{aligned}$$

is smooth, where e_j denotes the j-th vector of the standard basis of $\mathbb{R}^n$.

The *length* $l(c)$ of a smooth path[1] $c\colon [a,b] \to M$ is defined as

$$l(c) \coloneqq \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\mathrm{d}t$$

where $\dot{c}(t)$ is the tangent vector to the path c at the point $c(t)$.

If M is connected,

$$d(x,y) \coloneqq \inf\left\{l(c) : c \text{ a smooth path from } a \text{ to } b\right\}$$

defines the *Riemannian distance* between two points x and y.

A **geodesic** between two points is a smooth path that is length minimizing.

[1]By a smooth map $c\colon I \to M$ from an interval $I \subset \mathbb{R}$ into a smooth manifold we mean the restriction of a smooth map defined on an open interval containing I.

Definition: Isometry

An **isometry** between two Riemannian manifolds (M, g), (N, h) is a diffeomorphism $f\colon M \to N$ such that $g = f^*h$, that is, if $d_pf\colon T_pM \to T_{f(p)}N$ is the differential, then

$$h_{f(p)}(d_pf(u), d_pf(v)) = g_p(u, v)\,,$$

for all $u, v \in T_pM$.

It is easy to see that a Riemannian isometry maps geodesics to geodesics and hence preserves the Riemannian distance. But actually the converse also holds:

Theorem II.1: **[Hel01, Theorem I.11.1]**

Let M be a Riemannian manifold and $\varphi\colon M \to M$ a diffeomorphism. Then the following are equivalent:

(i) φ is a Riemannian isometry,

(ii) φ preserves the Riemannian distance.

The following is an extremely useful rigidity result for connected Riemannian manifolds. Its states that Riemannian isometries are completely determined by the local data at one point.

Lemma II.2: **[Hel01, Lemma I.11.2]**

Let $f_i\colon M \to N$, $i = 1, 2$, be two isometries between Riemannian manifolds and assume that M is connected. Suppose there exists a point $p \in M$ such that

$$f_1(p) = f_2(p) \qquad \text{and} \qquad d_pf_1 = d_pf_2.$$

Then $f_1 = f_2$.

We start the proof by recalling few facts that will be useful also in the following. The *Riemannian exponential map* Exp_p at a point $p \in M$ is defined from a neighborhood U_0 of $0 \in T_pM$ to a neighborhood of p in M as follows. Let $X_p \in T_pM$, and let γ_{X_p} be the unique geodesic $\gamma_{X_p}\colon (-\epsilon, \epsilon) \to M$, such that

$$\gamma_{X_p}(0) = p \qquad \text{and} \qquad \dot\gamma_{X_p}(0) = X_p\,.$$

Then

$$\operatorname{Exp}_p(X_p) := \gamma_{X_p}(1)\,.$$

An open neighborhood of p that is the diffeomorphic image of a star shaped neighborhood of $0 \in T_pM$ under Exp_p is called a **normal neighborhood**

If $f\colon M \to M$ is an isometry and $0 \in N_0 \subset T_pM$ (resp. $0 \in N_0' \subset T_{f(p)}M$) is a neighborhood where Exp_p (resp. $\mathrm{Exp}_{f(p)}$) is defined, then the diagram

$$\begin{array}{ccc} N_0 & \xrightarrow{d_pf} & N_0' \\ {\scriptstyle \mathrm{Exp}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{Exp}_{f(p)}} \\ M & \xrightarrow{f} & M \end{array}$$

commutes.

Proof of Lemma II.2 . By hypothesis $f \coloneqq f_2^{-1} \circ f_1\colon M \to M$ is an isometry that satisfies

$$f(p) = p \qquad \text{and} \qquad d_pf = Id\,.$$

so that the set

$$\mathcal{S} \coloneqq \{q \in M : f(q) = q,\ d_qf = Id\}$$

is closed and non-empty as $p \in \mathcal{S}$. We show that $\mathcal{S}$ is open.

Let $q \in \mathcal{S}$ and $U = \mathrm{Exp}_q(N_0)$ a normal neigborhood of q. Then for all $v \in T_qM$ and $t \in \mathbb{R}$ with $tv \in N_0$ we have

$$\begin{aligned} f(\mathrm{Exp}_q(tv)) &= \mathrm{Exp}_{f(q)}(td_qf(v)) \\ &= \mathrm{Exp}_q(td_qf(v)) \\ &= \mathrm{Exp}_q(tv) \end{aligned}$$

which shows that $f|_U = Id$. Thus $U \subset \mathcal{S}$ and hence $\mathcal{S}$ is open. As $\mathcal{S}$ is a closed and open non-empty set of the connected set M it is equal to all of M. ■

The isometries of a Riemannian manifold (M, g) form a group under composition, denoted $\mathrm{Iso}(M)$, that can be endowed with the compact-open topology, i.e. the topology generated by the subbasis

$$S(C, U) \coloneqq \{f \in \mathrm{Iso}(M) :\ f(C) \subset U\}$$

where $C \subset M$ is compact and $U \subset M$ is open.

Theorem II.3: **[Hel01, Theorem IV.2.5]**

Let M be a Riemannian manifold. Then $\mathrm{Iso}(M)$ with the compact-open topology is a locally compact group acting continuously on M.
Moreover, the stabiliser $\mathrm{Stab}_{\mathrm{Iso}(M)}(p)$ of a point $p \in M$ is compact.

Idea of proof. The proof relies upon the following two facts:

1. If M is a metric space, then the compact-open topology on $\mathrm{Iso}(M)$ coincides with the topology of uniform convergence on compact sets.

2. If $(f_n)_{n\geq 1} \subset M$ is a sequence such that for some $p \in M$ the sequence $(f_n(p))_{n\geq 1}$ converges, then there is $f \in \mathrm{Iso}(M)$ and a subsequence that converges to f in the compact-open topology.

To see the compactness of the stabiliser, we consider the map

$$\begin{aligned} \mathrm{Stab}_{\mathrm{Iso}(M)}(p) &\to O(T_pM) \\ f &\mapsto d_pf. \end{aligned}$$

Then Lemma II.2 implies that if $d_pf = Id$, then $f = Id$. ∎

II.2 Geodesic Symmetries

Definition: Riemannian Symmetric Spaces

Let M be a Riemannian manifold.

- M is **Riemannian locally symmetric** if for each $p \in M$ there exists a normal neighborhood U of p and an isometry $s_p\colon U \to U$ such that

  (1) $(s_p)^2 = Id$

  (2) p is an isolated fixed point, i.e. p is the only fixed point of s_p in U.

  The map $s_p\colon U \to U$ is called a **geodesic symmetry**.

- M is **Riemannian globally symmetric** if for each $p \in M$, s_p can be extended to an isometry defined on M.

Here is the relation between Riemannian locally symmetric and Riemannian symmetric spaces:

Theorem II.4: [Hel01, Theorem IV.5.6]

A complete simply connected Riemannian locally symmetric space is Riemannian globally symmetric.
In particular, the universal coveringof a complete locally symmetric space is globally symmetric and every complete locally symmetric space is a quotient of a complete globally symmetric space by a discrete torsion-free group of isometries isomorphic to the fundamental group.

Remark. The converse of Theorem II.4 does not hold, as for example S^1 is a Riemannian globally symmetric space that is Riemannian locally symmetric by definition but not simply connected.

We will only be concerned with Riemannian globally symmetric spaces, so the terminology "Riemannian symmetric space" or short RSS is from now on intended to mean "Riemannian globally symmetric space".

The following lemma relates the definition of geodesic symmetry at the beginningof this section with the intuitive one mentioned in the previous one.

Lemma II.5

Let M be a Riemannian manifold and $p \in U \subset M$ where $U = \text{Exp}(N_0)$ is a normal neighborhood of p. Let $s_p \in \text{Iso}(M)$ be an isometry such that p is the only fixed point. Then the following are equivalent:

(i) $(s_p)^2 = Id$

(ii) $d_p s_p = -Id$.

In either case, it holds that

$$s_p(\text{Exp}_p(tv)) = \text{Exp}_p(-tv)$$

wherever Exp *is defined.*

Proof. $(ii) \implies (i)$: By the chain rule it follows from $d_p s_p = -Id$ that

$$(d_p s_p)^2 = d_p(s_p)^2 = (-Id)^2 = Id = d_p Id.$$

Since $s_p^2(p) = p = Id(p)$ the claim follows from Lemma II.2.

$(i) \implies (ii)$: From $s_p^2 = Id$ we get $(d_p s_p)^2 = Id$, where $(d_p s_p)^2 \colon T_p(M) \to T_p(M)$. Hence $d_p s_p$ has eigenvalues [2] $+1$ or -1. If $+1$ were to be an eigenvalue, then there would be $0 \neq v \in T_p M$ such that $(d_p s_p)v = v$. Thus, for every $tv \in N_0$ we would have that

$$\begin{aligned} s_p(\text{Exp}(tv)) &= \text{Exp}(d_p s_p(tv)) \\ &= \text{Exp}(tv) \end{aligned}$$

Hence $\text{Exp}(tv)$ would be a fixed point for every t such that $tv \in N_0$, contradicting the fact that p is the only fixed point of s_p. ∎

[2] Let V be a real vector space. Then any map $A \in \text{End}(V)$ such that $A^2 = Id$ is diagonalizable. In fact, if $(\cdot,\cdot)$ is any inner product, then A is in the orthogonal group of the inner product $< u, v > := (u, v) + (Au, Av)$ and hence is diagonalizable.

The following corollary follows immediately from Lemma II.5 and Lemma II.2:

Corollary II.6

If M is a connected Riemannian manifold and $p \in M$, then there is at most one involutive isometry s_p with p as isolated fixed point.

Proposition II.7

If M is a Riemannian symmetric space, then it is complete. Moreover, the connected component $\mathrm{Iso}(M)^\circ$ of the isometry group $\mathrm{Iso}(M)$ acts transitively on M.

The completeness in Proposition II.7 is both as metric space and geodesically. This follows from the following classical theorem:

Theorem II.8: Hopf–Rinow

Let M be a connected Riemannian manifold. Then the following are equivalent:

(i) Closed and bounded sets are compact,

(ii) M is a complete metric space,

(iii) M is geodesically complete, that is, for all $p \in M$ the exponential map is defined on the whole tangent space.

As a consequence of any of the above, for all $p, q \in M$ there exists a geodesic connecting p and q.

The proof of Proposition II.7 relies on the following lemma, whose proof we postpone.

Lemma II.9

Let M be a Riemannian symmetric space. Then the map $M \to \mathrm{Iso}(M)$ defined by $p \mapsto s_p$ is continuous.

Remark. If M is a Riemannian symmetric space, $o \in M$ a basepoint and

$$K \coloneqq \mathrm{Stab}_{\mathrm{Iso}(M)}(o),$$

then the orbit map

$$\begin{aligned} \mathrm{Iso}(M)/K &\to M \\ gK &\mapsto g(o) \end{aligned}$$

is a homeomorphism.

Proof of Lemma II.9: We verify that

$$s_{g(p)} = g s_p g^{-1}, \tag{II.1}$$

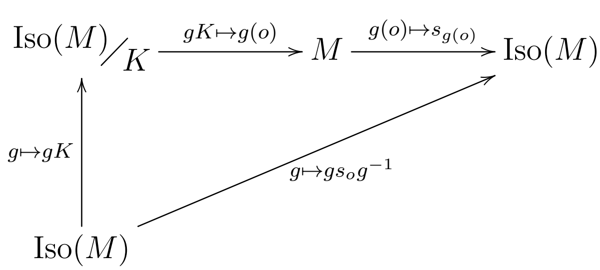
and, again by Lemma II.2 it is sufficient to check that the maps above and their differentials agree at some point:

$$\begin{aligned} g s_p g^{-1}(g(p)) &= g s_p(p) \\ &= g(p) \\ &= s_{g(p)}(g(p)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}_{g(p)}(g s_p g^{-1}) &= (\mathrm{d}_p g)(\mathrm{d}_p s_p)(\mathrm{d}_{g(p)} g^{-1}) \\ &= -(\mathrm{d}_p g)(\mathrm{d}_{g(p)} g^{-1}) \\ &= -\mathrm{d}_{g(p)} Id \\ &= -Id \\ &= \mathrm{d}_{g(p)} s_{g(p)}. \end{aligned}$$

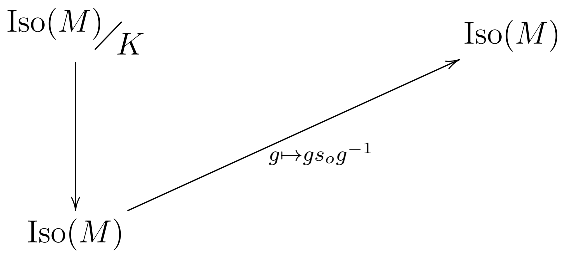
Let $p \in M$ and let $g \in \mathrm{Iso}(M)$ be such that $g(o) = p$. Consider then the following diagram



where:

(1) The first arrow in the top line $\mathrm{Iso}(M)/K \to M$ is the orbit map $gK \mapsto g(o)$, which is a homeomorphism.

(2) The diagonal arrow $\mathrm{Iso}(M) \to \mathrm{Iso}(M)$ defined as $g \mapsto g s_o g^{-1} = s_{g(o)}$ is continuous because of (II.2) and since $\mathrm{Iso}(M)$ is a topological group. Moreover it factors through K, since $K = \mathrm{Stab}_{\mathrm{Iso}(M)}(o)$, thus giving a continuous map

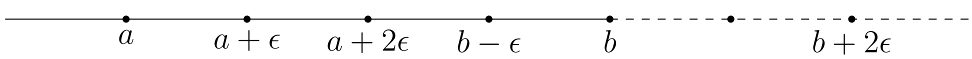


(II.2)

The composition of the inverse of the orbit map with the map in (II.2) realizes $M \to \mathrm{Iso}(M)$, $g(o) \mapsto s_{g(o)}$ as composition of continuous maps.

■

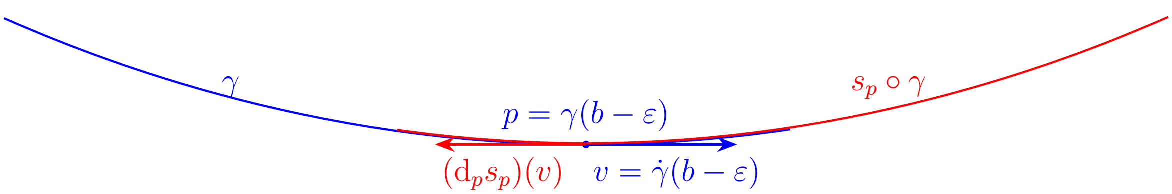
Proof of Proposition II.7: Let $a < b$ and $\gamma\colon (a,b) \to M$ a geodesic segment. We will show that γ can be extended to $(a, b+2\varepsilon)$ where $\varepsilon \coloneqq \frac{b-a}{4}$.



This will show that γ can be extended to $\mathbb{R}$ and hence M is geodesically complete and, by Theorem II.8, also metrically complete.

Let $p \coloneqq \gamma(b-\varepsilon)$ and consider the geodesic segment

$$\begin{aligned} \eta\colon\ (a+2\varepsilon, b+2\varepsilon) &\longrightarrow M \\ t &\mapsto s_p\left(\gamma\left(a+b+2\varepsilon-t\right)\right) . \end{aligned}$$



Note that this makes sense, because $t \in (a+2\varepsilon, b+2\varepsilon)$ implies that

$$a = a+b+2\varepsilon-(b+2\varepsilon) < a+b+2\varepsilon-t < a+b+2\varepsilon-(a+2\varepsilon) = b\,.$$

To see that η extends γ as geodesica, we need to check that

(1) $\eta(b-\epsilon) = \gamma(b-\epsilon)$, and

(2) $\dot\eta(b-\epsilon) = \dot\gamma(b-\epsilon)\,.$

In fact,

$$\eta(b-\varepsilon)=\eta(a+3\varepsilon)=s_p\left(\gamma(b-\varepsilon)\right)=\gamma(b-\varepsilon)$$

since s_p fixes p. Also, by the chain rule,

$$\begin{aligned}
\dot\eta\,(b-\varepsilon) &= \dot\eta\,(a+3\varepsilon)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\eta\,(a+3\varepsilon+t)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}s_p\left(\gamma\left(a+b+2\epsilon-(a+3\epsilon+t)\right)\right)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}s_p\left(\gamma\,(b-\varepsilon-t)\right)\\
&= (\mathrm{d}_p s_p)\left(-\dot\gamma\,(b-\varepsilon)\right)\\
&= \dot\gamma\,(b-\varepsilon)
\end{aligned}$$

which implies by the uniqueness of geodesics that

$$\eta|_{(b-2\varepsilon,b)}=\gamma|_{(b-2\varepsilon,b)}\,.$$

This means that we can prolong γ to $(a,b+2\varepsilon)$ using η.

Let $p,q\in M$ and $\gamma\colon[0,t]\to M$ with $\gamma(0)=p,\ \gamma(t)=q$. Then

$$q=s_{\gamma(t/2)}(p)\,,$$

which shows that $\mathrm{Iso}(M)$ acts transitively on M. We want however to show that $\mathrm{Iso}(M)^\circ$ acts transitively. Since we showed in Lemma II.9 that the map $p\mapsto s_p$ is continuous and we know that M is connected, the image of M is contained in a connected component, although there is no guarantee that it is the connected component of the identity. We consider then the map

$$\begin{aligned}
M\times M &\to \mathrm{Iso}(M)\\
(p,q) &\mapsto s_p\circ s_q
\end{aligned}$$

which is continuous and whose image contains $s_p^2=Id$. It follows then that the image of this map is contained in $\mathrm{Iso}(M)^\circ$. If γ is a geodesic from p to q with $\gamma(0)=p$ and $\gamma(t)=q$, then

$$s_{\gamma(t/2)}\circ s_p(p)=q\,.$$

■

Corollary II.10

Let (M,g) be a Riemannian symmetric space, $p\in M$ and $K=\mathrm{Stab}_{\mathrm{Iso}(M)}(p)$. Then K meets every connected component of $\mathrm{Iso}(M)$.
In particular, $\mathrm{Iso}(M)^\circ$ is open and of finite index in $\mathrm{Iso}(M)$.

Proof. Take $g \in \mathrm{Iso}(M)$. Since $\mathrm{Iso}(M)^\circ$ acts transitively, there is $g_0 \in \mathrm{Iso}(M)^\circ$ such that $g_0 p = gp$. But that means that there is an element $k \in K$ such that $g = g_0 k$. Thus $k \in g\,\mathrm{Iso}(M)^\circ$ and K meets every connected component.

To see the second assertion, we note that $Id \in K^\circ$. Thus $K^\circ \leq \mathrm{Iso}(M)^\circ$ and the homomorphism

$$\alpha\colon K \to {}^{\mathrm{Iso}(M)}\!/\!_{\mathrm{Iso}(M)^\circ}$$

factors through K°:

$$K/K^\circ \to \mathrm{Iso}(M)/\mathrm{Iso}(M)^\circ .$$

By the first assertion this is surjective and hence

$$\left| K/K^\circ \right| < \infty \implies \left| \mathrm{Iso}(M)/\mathrm{Iso}(M)^\circ \right| < \infty \qquad \blacksquare$$

A classical theorem of Myers and Steenrod [MS39] asserts that the isometry group of a Riemannian manifold is a Lie group. The idea is to consider orbits of points and parametrise in this way $\mathrm{Iso}(M)$. We sketch here the proof in the special case of our interest: namely, knowing that a RSS is a homogeneous space G/K we consider the principal G-bundle $G \twoheadrightarrow G/K$ and we induce the Lie group structure as a local product, using that $K \hookrightarrow \mathrm{O}(n, \mathbb{R})$ is a Lie group.

Theorem II.11: **[Hel01, Lemma IV.3.2, Theorem 3.3 (i)]**

Let M be a Riemannian symmetric space. Then $G \coloneqq \mathrm{Iso}(M)$ has a Lie group structure compatible with the compact-open topology and it acts smoothly on M.
Moreover, if $o \in M$ is a base point, then M is diffeomorphic to G/K, where $K = \mathrm{Stab}_G(o)$ and contains no non-trivial normal subgroups of G.

Sketch of the proof. The map $K \to \mathrm{O}(T_oM, g)$, defined by $k \mapsto \mathrm{d}_o k$, is a homeomorphism onto its image. Hence K can be identified with a closed subgroup of $\mathrm{O}(T_oM, g)$, from which it inherits a unique differentiable structure compatible with the topology, which makes it a Lie group.

Let $\pi\colon G \twoheadrightarrow M = G/K$ be the natural projection, $\pi(g) \coloneqq g(o)$. We will construct a continuous local section of π, that is a map $\varphi\colon U \to G$, where U is a normal neighborhood of p in M, such that $\pi \circ \varphi = Id$. From this it will follow that φ is a homeomorphism onto its image $B \coloneqq \varphi(U)$ (it is clearly injective and its continuous inverse is $\pi|_B$). Thus we can define

$$\begin{aligned} \tilde{\varphi}\colon U \times K &\to \pi^{-1}(U) \\ (x, k) &\mapsto \varphi(x)k \end{aligned}$$

that is continuous and bijective with inverse map given by

$$\begin{aligned} \tilde{\varphi}^{-1}\colon \pi^{-1}(U) \to & \quad U \times K \\ g \quad & \mapsto (g(p), \varphi(g(p))p). \end{aligned}$$

Thus $\tilde{\varphi}^{-1}$ is a homeomorphism between $\pi^{-1}(U) \ni Id$ and $U \times K$. The smooth structure on G is then given by the smooth structure on translates of $\pi^{-1}(U)$. The differentiable structure will hence be given to G by using translates of open set BU, where $U \subset K$ is open and one can check that all the needed properties hold.

In order to construct the section φ, let $\gamma(t)$ be a geodesic in U such that $\gamma(0) = p$. As seen already in the proof of Proposition II.7, for every t, the isometry $s_{\gamma(t/2)} \circ s_p$ maps p into $\gamma(t)$, that is

$$s_{\gamma(t/2)} \circ s_p(p) = \gamma(t).$$

Define then $\varphi(\gamma(t)) \coloneqq s_{\gamma(t/2)} \circ s_o$. The map φ has the desired properties, since it is obviously injective for small enough t and continuous (Lemma II.9).

If K were to contain a subgroup that is normal in G, then this subgroup would act trivially on $M = G/K$, which is impossible. ■

II.3 Concepts of Riemannian Geometry

Definition: Vector fields

Let M be a smooth manifold, $\pi\colon TM \to M$ be the tangent bundle. A smooth vector field is a section of π, that is a map $X\colon M \to TM$ such that $\pi \circ X = Id_M$.

We denote by $\mathrm{Vect}(M)$ the set of vector fields, which is a $C^\infty(M)$-module with pointwise multiplication

$$(fX)_p = f(p)X_p \qquad \text{for} \qquad f \in C^\infty(M), X \in \mathrm{Vect}(M).$$

If $f \in C^\infty(M, M)$, we denote by $\mathrm{d}_p f\colon T_pM \to T_{f(p)}M$ its differential. Then any $X \in \mathrm{Vect}(M)$ acts on $C^\infty(M, M)$ by

$$(Xf)(p) = (\mathrm{d}_p f)(X_p)$$

At a point $m \in M$, this amounts to taking the derivative of f in the direction of X_p.

While functions can be differentiated on a manifold, we need a canonical way of identifying tangent spaces at different points if we want to differentiate vector fields. This is exactly what is achieved with a connection.

Definition: Connection

An **affine connection** on M is a map

$$\nabla\colon \operatorname{Vect}(M) \times \operatorname{Vect}(M) \to \operatorname{Vect}(M)$$

such that for all $X, X', Y, Y' \in \operatorname{Vect}(M)$, for all $f, f' \in C^\infty(M)$ and for all $a, b \in \mathbb{R}$,

(1) ∇ is $C^\infty(M)$-linear in the first variable, that is

$$\nabla_{fX+f'X'}(Y) = f\nabla_X Y + f'\nabla_X Y$$

(2) ∇ is $\mathbb{R}$-linear in the second variable, that is

$$\nabla_X(aY + bY') = a\nabla_X Y + b\nabla_X Y'$$

(3) ∇ satisfies the Leibniz-rule, that is

$$\nabla_X(fY + f'Y') = f\nabla_X Y + f'\nabla_X Y' + (Xf)Y + (Xf')Y'\,.$$

Remark. The connection $\nabla_X Y(p)$ amounts to taking the derivative at $p \in M$ of Y in the direction of X_p. In fact, the value at the point $p \in M$ of $\nabla_X Y$ depends only on the value X_p of the vector field X at p, but on the other hand on the vector field Y in a neighborhood of p.

Definition: Covariant Derivative

Let $\gamma\colon I \to M$ be a smooth curve. A **vector field along** γ is a smooth map $X\colon I \to TM$ such that $X(t) \in T_{\gamma(t)}M$. The **covariant derivative** of a vector field X along γ is $\nabla_{\dot\gamma(t)}X$.

We write $\operatorname{Vect}(\gamma^*TM)$ for the vector space of vector fields along γ. Note that a vector field along γ is only a vector field whose basepoint is on γ, but not necessarily tangent to γ.

Definition: Parallel Vector Fields

Let $X \in \operatorname{Vect}(\gamma^*TM)$ be a vector field along a smooth curve γ. We say that X is **parallel** if $\nabla_{\dot\gamma}X = 0$.

Remark. Take $\gamma \subset \mathbb{R}^n$ and $X \in \operatorname{Vect}(\gamma^*TM)$. We can decompose

$$T\mathbb{R}^n = \mathbb{R}\dot\gamma \oplus (\mathbb{R}\dot\gamma)^\perp$$

Then it holds that

$$\nabla_{\dot\gamma} X = \mathrm{pr}_{(\mathbb{R}\dot\gamma)^\perp}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)$$

The same applies to the tangent vector along a great circle in $S^{n-1} \subset \mathbb{R}^n$ parametrised by arclength. In fact, geodesics can be defined as curves γ such that

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

■

In general it is extremely rare to find "constant" vector fields, that is vector fields $Y \in \mathrm{Vect}(M)$ such that

$$(\nabla_X Y)_p = 0\,. \tag{II.3}$$

for any $p \in M$ and all $X \in \mathrm{Vect}(M)$. This is because the equation (II.3) is an overdetermined partial differential equation. On the other hand the existence and uniqueness of the solutions of differential equations imply the following:

Proposition II.12

Let M be a differential manifold and $\gamma \in M$ a smooth curve. Given $v \in T_{\gamma(0)}M$, there is a unique vector field $X^v \in \mathrm{Vect}(\gamma^*TM)$ parallel along γ and such that $X^v_{\gamma(0)} = v$.

Definition: Parallel Transport

We can then define the **parallel transport along a curve** γ from $\gamma(0)$ to $\gamma(t)$ as

$$\begin{aligned}\mathbb{P}_{\gamma,[0,t]} : T_{\gamma(0)}M &\to T_{\gamma(t)}M\\ v &\mapsto X^v_{\gamma(t)}\end{aligned}$$

Because of uniqueness,

$$\mathbb{P}_{\gamma,[t_1,t_2]} \circ \mathbb{P}_{\gamma,[t_0,t_1]} = \mathbb{P}_{\gamma,[t_0,t_2]}\,.$$

Vector fiels are locally differential operators of first order. It is hence clear that the composition of two differential operators in general is not anymore a vector field. This leads to the definition of the bracket [,] of two vector fields. If $f \in C^\infty(M)$, $X, Y \in \mathrm{Vect}(M)$ and $p \in M$, then

$$[X,Y](f)(p) := X_p(Yf) - Y_p(XF)\,.$$

If our Riemannian manifold has a Riemannian structure, one wants that an affine connection is compatible with the Riemannian structure. This leads to the following:

Definition: Riemannian Connection

Let (M, g) be a Riemannian manifold. A **Riemannian connection** on (M, g) is an affine connection such that in addition for every $X, Y \in \mathrm{Vect}(M)$

(4) $\nabla_X Y - \nabla_Y X = [X, Y]$

(5) $Xg(Y, Y') = g(\nabla_X Y, Y') + g(Y, \nabla_X Y')$

Remark. If $\gamma\colon I \to M$ is a smooth curve, then the last condition can also be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} g(Y, Y')_{\gamma(t)} = g(\nabla_{\dot\gamma} Y, Y')_{\gamma(t)} + g(Y, \nabla_{\dot\gamma} Y')_{\gamma(t)}$$

In particular if Y, Y' are parallel vector fields along γ, then

$$\nabla_{\dot\gamma} Y = \nabla_{\dot\gamma} Y' = 0$$

implies that $g(Y, Y')_{\gamma(t)}$ is constant with respect to t. Thus parallel transport preserves the inner product.

Theorem II.13: Fundamental Theorem in Riemannian Geometry

Given a Riemannian manifold (M, g), there exists a unique Riemannian connection called the **Levi-Civita connection***.*

Recall that a diffeomorphism $f\colon M \to M$ induces a linear map on vector fields via the **pushforward**:

$$(f_* X)_p = \mathrm{d}_{f^{-1}(p)} X_{f^{-1}(p)}$$

which preserves the bracket

$$f_*([X, Y]) = [f_* X, f_* Y]\,.$$

Lemma II.14

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve and $Y \in \mathrm{Vect}(\gamma^* TM)$ a parallel vector field. If $f \in \mathrm{Iso}(M)$, then $f_* Y$ is a parallel vector field along $f \circ \gamma$.

Proof. We define

$$\begin{aligned} D\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) &\longrightarrow \mathrm{Vect}(M) \\ (X, Y) &\mapsto f_*^{-1}(\nabla_{f_* X} f_* Y) \coloneqq D_X Y \end{aligned}$$

and show that all five properties of a Riemannian connection are satisfied. It then follows by uniqueness that

$$\nabla_X Y = f_*^{-1}(\nabla_{f_*X} f_*Y) \implies f_*(\nabla_X Y) = \nabla_{f_*X} f_*Y.$$

If now $X = \dot\gamma$, then

$$f_*(\underbrace{\nabla_{\dot\gamma} Y}_{=0}) = \nabla_{f_*\dot\gamma} f_*Y = 0$$

which means that f_*Y is parallel along $f_*\gamma = f \circ \gamma$.

While the properties (1), (2), (3) are obvious, we have to verify the last two:

(4)
$$\begin{aligned} D_X Y - D_Y X &\overset{\text{def}}{=} f_*^{-1}(\nabla_{f_*X} f_*Y) - f_*^{-1}(\nabla_{f_*Y} f_*X) \\ &\overset{f_* \text{ linear}}{=} f_*^{-1}(\nabla_{f_*X} f_*Y - \nabla_{f_*Y} f_*X) \\ &\overset{(4) \text{ of } \nabla}{=} f_*^{-1}([f_*X, f_*Y]) \\ &\overset{f_* \text{ Lie alg. homo.}}{=} [X, Y] \end{aligned}$$

(5)
$$\begin{aligned} g(D_X Y, Y') + g(Y, D_X Y') &\overset{\text{def}}{=} g(f_*^{-1}\nabla_{f_*X} Y, Y') + g(Y, f_*^{-1}\nabla_{f_*X} f_*Y') \\ &\overset{f \in \mathrm{Iso}(M)}{=} g(\nabla_{f_*X} f_*Y, f_*Y') + g(f_*Y, \nabla_{f_*X} f_*Y') \\ &\overset{(5) \text{ of } \nabla}{=} (f_*X) g(f_*Y, f_*Y') \\ &= X g(Y, Y'). \end{aligned}$$

■

Remark. Diff(M) acts on the set of affine connections as follows: Let $f\colon M \to M$ be a Diffeomorphism and $\nabla\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ be a connection. Then

$$\begin{aligned} D\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) &\longrightarrow \qquad \mathrm{Vect}(M) \\ (X, Y) \qquad &\mapsto f_*^{-1}(\nabla_{f_*X} f_*Y) := D_X Y \end{aligned}$$

is also an affine connection.

In particular, if $M = G$ and $f = L_g$ we say that ∇ is left-invariant if

$$\nabla_X Y = (L_g)_*^{-1}(\nabla_{(L_g)_*X}(L_g)_*Y).$$

II.4 Transvections and Parallel Transport

We saw in the proof of Proposition II.7 that the set of geodesic symmetries is transitive on a Riemannian globally symmetric space. In particular, we saw that if

$p, q \in M$ and $\gamma \colon \mathbb{R} \to M$ is geodesic such that $\gamma(0) = p$ and $\gamma(t) = q$, then $q = s_{\gamma(t/2)} \circ s_{\gamma(0)}(p)$.

Definition: Transvections

The isometry $\mathcal{T}_{\gamma,t} := s_{\gamma(t/2)} \circ s_{\gamma(0)}$ is called **transvection along** γ.

The first assertion of the following proposition explains the reason for this terminology.

Proposition II.15

Let M be a Riemannian globally symmetric space, $\gamma \colon \mathbb{R} \to M$ a geodesic and $\mathcal{T}_{\gamma,t} := s_{\gamma(t/2)} \circ s_{\gamma(0)}$ the associated transvection.

(1) For every $c \in \mathbb{R}$,

$$\mathcal{T}_{\gamma,t}(\gamma(c)) = \gamma(t + c)\,.$$

(2) The differential $\mathrm{d}_{\gamma(0)}\mathcal{T}_{\gamma,t} \colon T_{\gamma(0)}M \to T_{\gamma(t)}M$ is the parallel translation along the geodesic γ, that is, if $v = X_{\gamma(0)} \in T_{\gamma(0)}M$, then $\mathrm{d}_{\gamma(0)}\mathcal{T}_{\gamma,t}v$ is the associated parallel vector field along γ, i.e.

$$(\mathrm{d}_{\gamma(0)}\mathcal{T}_{\gamma,t})(X^v)_{\gamma(0)} = (X^v)_{\gamma(t)} \tag{II.4}$$

(3) The map $t \mapsto \mathcal{T}_{\gamma,t}$ is a one-parameter group in $\mathrm{Iso}(M)^\circ$.

(4) $\mathcal{T}_{\gamma,t}$ is independent on the parametrisation of γ.

Proof. (1) Since geodesic symmetries map geodesics onto themselves changing the orientation, the map $\mathcal{T}_{\gamma,t}$ must map the geodesic γ onto itself and preserve its orientation. If we assume that γ is a unit speed parametrization, it follows that the restriction to the geodesic $\gamma(t)$ has the form $\mathcal{T}_{\gamma,t}(\gamma(c)) = \gamma(c + \text{constant})$. Since $\mathcal{T}_{\gamma,t}(\gamma(0)) = \gamma(t)$, then $\mathcal{T}_{\gamma,t}(\gamma(c)) = \gamma(t + c)$.

(2) Using the definition of $\mathcal{T}_{\gamma,t}$ and the chain rule we get from the left hand side of (II.4)

$$\begin{aligned}(\mathrm{d}_{\gamma(0)}\mathcal{T}_{\gamma,t})(X^v)_{\gamma(0)} &= \mathrm{d}_{\gamma(0)}(s_{\gamma(t/2)} \circ s_{\gamma(0)})(X^v)_{\gamma(0)} \\ &= (\mathrm{d}_{\gamma(0)}s_{\gamma(t/2)})(\mathrm{d}_{\gamma(0)}s_{\gamma(0)})(X^v)_{\gamma(0)} \\ &= (\mathrm{d}_{\gamma(0)}s_{\gamma(t/2)}(X^v)_{\gamma(0)}\,.\end{aligned}$$

To elaborate on the right hand side of (II.4) we start with the following:

Claim: For every $\ell \in \mathbb{R}$,

$$(s_{\gamma(\ell)})_* X^v = -X^v \,. \tag{II.5}$$

In fact, since $s_{\gamma(l)}$ is an isometry for every l and X^v is parallel along γ, then by Lemma II.14 $(s_{\gamma(l)})_* X^v$ is a vector field parallel along $s_{\gamma(l)} \circ \gamma = \gamma$. At the point $\gamma(l)$ the value of this new parallel vector field is

$$\begin{aligned}
(s_{\gamma(l)})_*(X^v)_{\gamma(l)} &= \mathrm{d}_{s^{-1}_{\gamma(l)}(\gamma(l))} s_{\gamma(l)} X_{s^{-1}_{\gamma(l)}(\gamma(l))} \\
&= \underbrace{(\mathrm{d}_{\gamma(l)} s_{\gamma(l)})}_{=-Id} X^v_{\gamma(l)} \\
&= -(X^v)_{\gamma(l)}
\end{aligned}$$

But $-X^v$ is also parallel along γ with value $-(X^v)_{\gamma(l)}$ at $\gamma(l)$. By uniqueness of parallel vector fields with prescribed initial conditions we have proven the claim.

Because of the claim with $\ell = t/2$ and using the definition of the pushforward, the right hand side of (II.4) becomes

$$\begin{aligned}
(X^v)_{\gamma(t)} &= -(s_{\gamma(t/2)})_*(X^v)_{\gamma(t)} \\
&= -\left(\mathrm{d}_{s^{-1}_{\gamma(t/2)}(\gamma(t))} s_{\gamma(t/2)}\right) (X^v)_{s^{-1}_{\gamma(t/2)}(\gamma(t))} \\
&= -\left(\mathrm{d}_{\gamma(0)} s_{\gamma(t/2)}\right) (X^v)_{\gamma(0)} \,,
\end{aligned}$$

which concludes the proof.

(3) This follows from (1), (2) and from the fact that parallel transport is a one-parameter subgroup. In fact $\mathcal{T}_{\gamma,t_1+t_2}(\gamma(c)) = \mathcal{T}_{\gamma,t_2} \circ \mathcal{T}_{\gamma,t_1}(\gamma(c))$ and furthermore

$$\begin{aligned}
\mathrm{d}_{\gamma(t)} \mathcal{T}_{\gamma,t_1+t_2} &= \mathbb{P}_{\gamma,[c,t_1+t_2+c]} \\
&= \mathbb{P}_{\gamma,[c+t_1,c+t_1+t_2]} \circ \mathbb{P}_{\gamma,[c,c+t_1]} \\
&= (\mathrm{d}_{\gamma(c+t_1)} \mathcal{T}_{\gamma,t_2}) \circ (\mathrm{d}_{\gamma(c)} \mathcal{T}_{\gamma,t_1}) \\
&= \mathrm{d}_{\gamma(c)} (\mathcal{T}_{\gamma,t_2} \circ \mathcal{T}_{\gamma,t_1}).
\end{aligned}$$

we conclude by Lemma II.2 that $\mathcal{T}_{\gamma,t_1+t_2} = \mathcal{T}_{\gamma,t_2} \circ \mathcal{T}_{\gamma,t_1}$.

(4) **A unit speed reparametrisation** of γ is $t \mapsto t + a$. Thus

$$\begin{aligned}
s_{\gamma(t/2+a)} s_{\gamma(a)} &= s_{\gamma(t/2+a)} s_{\gamma(0)} s_{\gamma(0)} s_{\gamma(a)} \\
&= \mathcal{T}_{\gamma,t+2a} (s_{\gamma(a)} s_{\gamma(0)})^{-1} \\
&= \mathcal{T}_{\gamma,t+2a} (\mathcal{T}_{\gamma,2a})^{-1} \\
&= \mathcal{T}_{\gamma,t}
\end{aligned}$$

■

Definition: One-parameter Group of Transvections

The map

$$\begin{aligned} \mathbb{R} &\to \mathrm{Iso}(M)^\circ \\ t &\mapsto \quad \mathcal{T}_{\gamma,t} \end{aligned}$$

is called a **one-parameter group of transvections** associated to the geodesic γ.

II.5 Algebraic Point of View

We have seen that if M is Riemannian (globally) symmetric, then M is diffeomorphic to G/K, where $G = \mathrm{Iso}(M)^\circ$ and K is the stabiliser of a point in M. In this section we will deal with the natural question regarding the converse statement: namely, which homogeneous spaces are Riemannian symmetric spaces?

Definition: Involution

A Lie group automorphism $\sigma\colon G \to G$ is an **involution** if $\sigma^2 = Id$ and $\sigma \neq Id_G$.

If $\sigma \in \mathrm{Aut}(G)$, we set $G^\sigma := \{g \in G : \sigma(g) = g\}$.

Proposition II.16

Let M be a Riemannian symmetric space and $G := \mathrm{Iso}(M)^\circ$. Fix a base point $o \in M$ and let $K = \mathrm{Stab}_G(o)$ be the isotropy subgroup of G at o. Then the automorphism

$$\begin{aligned} \sigma\colon G &\to \quad G \\ g &\mapsto s_o g s_o \end{aligned}$$

is an involution of G and

$$(G^\sigma)^\circ \leq K \leq G^\sigma.$$

Proof. First we verify that $g \mapsto s_o g s_o$ is involutive. In fact, since s_o^2 is the identity,

$$\sigma^2(g) = \sigma(\sigma(g)) = \sigma(s_o g s_o) = s_o(s_o g s_o)s_o = s_o^2 g s_o^2 = g\,.$$

We verify now that $K \leq G^\sigma$, that is that for every $k \in K$, $\sigma(k) = s_o k s_o = k$. To see this observe first of all that

$$\sigma(k)(o) = (s_o k s_o)(o) = s_o(k(s_o(o))) = s_o(k(o)) = s_o(o) = o.$$

Moreover, as $\mathrm{d}_o\sigma(k)\colon T_oM \to T_oM$ and $\mathrm{d}_o s_o = -Id$, we have that

$$\mathrm{d}_o\sigma(k) = \mathrm{d}_o(s_o k s_o) = (\mathrm{d}_o s_o)(\mathrm{d}_o k)(\mathrm{d}_o s_o) = \mathrm{d}_o k.$$

By the usual rigidity argument of Lemma II.2, $\sigma(k) = k$, that is $K \leq G^\sigma$.

Conversely, to show that $(G^\sigma)^\circ \leq K$, it is enough to see that K contains a neighborhood of the identity in G^σ. Let $V \subset M$ be an open neighborhood of $o \in M$. By continuity of the $G-$action on M, there exists an open neighborhood $U \subset G^\sigma$ of e such that $g(o) \in V$ for all $g \in U$. But if $g \in U \subset G^\sigma$, then

$$g = \sigma(g) = s_o g s_o,$$

so that $g(o) \in V$ is a fixed point of s_o as

$$s_o g(o) = g s_o(o) = g(o).$$

Since s_o has only isolated fixed points, we could chose V small enough such that o is the only fixed point of s_o in V, which implies that $g(o) = o$. Thus $U \subset K$. ■

Notice that one cannot say anything more precise of the relation between K and G^σ, as the following examples show:

Example. (1) Let $M = S^2$, $o = e_3$ and $G = \mathrm{Iso}(M) = \mathrm{SO}(3,\mathbb{R})$. We can write s_o and $g \in \mathrm{SO}(3,\mathbb{R})$ in block form as

$$s_o = \begin{pmatrix} -Id_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} A & b \\ c & d \end{pmatrix},$$

so that

$$\sigma(g) = \begin{pmatrix} -Id_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} -Id_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}.$$

Thus

$$G^\sigma = \left\{ g \in \mathrm{SO}(3,\mathbb{R}) : \; g = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \text{ with } A \in \mathrm{O}(2,\mathbb{R}), d = \pm 1, (\det A)d = 1 \right\}$$

has two connected components. Since S^2 is simply connected and G is connected, then also K is connected[3], so that $(G^\sigma)^\circ = K \lneq G^\sigma$ and

$$K = \left\{ g \in \mathrm{SO}(3,\mathbb{R}) : \; g = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \text{ with } A \in \mathrm{SO}(2,\mathbb{R}) \right\}.$$

[3]If G is a connected topological group, $H \leq G$ a closed subgroup such that G/H is simply connected, then H is connected. In fact, let H° be the connected component of the identity of H. Then $G/H^\circ \twoheadrightarrow G/H$ is a covering map. Moreover, since G is connected, then G/H is connected. Since G/H is simply connected, the covering map must be the identity.

(2) If $M = \mathbb{P}(\mathbb{R}^3) = S^2/\{\pm Id\}$, then $G = \mathrm{Iso}(M)^\circ = \mathrm{Iso}(M) = \mathrm{O}(3,\mathbb{R})/\{\pm Id\}$. Since $S^2 \twoheadrightarrow \mathbb{P}(\mathbb{R}^3)$, any isometry of $\mathbb{P}(\mathbb{R}^3)$ lifts to an isometry of S^2. If $g \in \mathrm{Stab}_G([e_3])$ then $g([e_3]) = [g(e_3)] = [e_3]$, so that $g = \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$. Thus $\mathrm{Stab}_G([e_3]) = (\mathrm{O}(2,\mathbb{R}) \times \mathrm{O}(1,\mathbb{R}))\,/ \pm Id$, which has two connected components. In this case also $\sigma\colon \mathrm{Iso}(M) \to \mathrm{Iso}(M)$ is $\sigma \begin{pmatrix} A & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}$, since it commutes with $\pm Id$, so that $G^\sigma = (\mathrm{O}(2,\mathbb{R}) \times \mathrm{O}(1,\mathbb{R}))\,/ \pm Id$. Thus $(G^\sigma)^\circ \neq K = G^\sigma$.

We point out that the phenomena arising in these examples can occur only in symmetric spaces that are compact. In fact a non-compact symmetric space is contractible and hence in particular simply connected. As a consequence the covering map $G/K^\circ \twoheadrightarrow G/K$ must be the identity and hence K must be connected.

Definition: Riemannian Symmetric Pair

Let G be a connected Lie group and $K \leq G$ a closed subgroup. The pair (G, K) is called **a Riemannian symmetric pair** if:

(1) $\mathrm{Ad}_{\mathrm{G}}(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, and

(2) There exists an involutive automorphism $\sigma \in \mathrm{Aut}(G)$ of G such that

$$(G^\sigma)^\circ \leq K \leq G^\sigma\,.$$

Remark. Proposition II.16 shows that a Riemannian symmetric space yields a Riemannian symmetric pair. It makes sense to give then the following definition:

Definition: Riemannian Symmetric Pair associated to (M, o)

Let M be a Riemannian (globally) symmetric space, $G = \mathrm{Iso}(M)^\circ$ and $K \leq G$ the isotropy subgroup of a point $o \in M$. Then (G, K) is called the **Riemannian symmetric pair associated to** (M, o).

The following theorem answers in particular the question at the beginningof this section.

Theorem II.17

Let (G, K) be a Riemannian symmetric pair with an involutive automorphism σ of G. Then G/K is a Riemannian symmetric space with respect to any G-invariant Riemannian metric.
If $\pi\colon G \to G/K$ denotes the natural projection, s_o the geodesic symmetry at $o = \pi(K) = eK \in G/K$, is defined by the equation

$$s_o \circ \pi = \pi \circ \sigma\,. \tag{II.6}$$

Corollary II.18

The geodesic symmetry s_o is independent of the choice of the G-invariant Riemannian metric on M.

Remark. Recall that $\ker(\mathrm{Ad}) = Z(G)$ and

$$K\big/K \cap Z(G) \xrightarrow{\cong} \mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$$

so that, loosely speaking, "the hypotheses is a bit less rigid than K being compact as the center might compensate for some non-compactness."

Example. Let $G := \widetilde{\mathrm{SL}(2,\mathbb{R})}$ be the universal coveringof $\mathrm{SL}(2,\mathbb{R})$, and let $\sigma\colon\ \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ be defined by $\sigma(g) := {}^t g^{-1}$. Let then $\tilde{\sigma}\colon G \to G$ be the unique lift of σ and $p\colon G \twoheadrightarrow SL(2,\mathbb{R})$ be the covering map. Then

$$G^{\tilde{\sigma}} = p^{-1}(\mathrm{SO}(2,\mathbb{R})) \cong \mathbb{R}$$

is not compact but

$$\mathrm{Ad}_G(G^{\sigma}) \cong \mathrm{SL}(2,\mathbb{R})\big/\pm Id$$

is compact.

Before we prove Theorem II.17, we have a look at some examples of Riemannian symmetric pairs:

Example. (1) $G < \mathrm{GL}(n,\mathbb{R})$ closed under transposition (e.g. $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$ of $\mathrm{SO}(p,q)^{\circ}$). Let $\sigma \in \mathrm{Aut}(G)$ be

$$\sigma(g) = {}^t g^{-1}$$

If G is not a subgroup of $\mathrm{O}(n,\mathbb{R})$, then σ is an involution, $G^{\sigma} = G \cap \mathrm{O}(n,\mathbb{R})$ and G^{σ} is connected such that (G, G^{σ}) is a Riemannian symmetric pair.

(2) Let $G < \mathrm{GL}(n,\mathbb{C})$ be a closed connected subgroup which is invariant under $g \overset{\sigma}{\mapsto} g^* = {}^t\bar{g}$. If G is not a subgroup of the unitary group $\mathrm{U}(n)$, then σ is an involution, $G^\sigma = G \cap \mathrm{U}(n)$ is connected and (G, G^σ) is a Riemannian symmetric pair. One could take for instance $G = \mathrm{SL}(n,\mathbb{C}), \mathrm{GL}(n,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C})$.

(3) Take $G = \mathrm{SO}(n,\mathbb{R})$, $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ and $r \in \mathrm{SO}(n,\mathbb{R})$ such that $r|_{\mathbb{R}^p} = Id_p$ and $r|_{\mathbb{R}^q} = -Id_q$:

$$r = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_q \end{pmatrix}$$

Then

$$G^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{O}(p), B \in \mathrm{O}(q), \det(A)\det(B) = 1 \right\}$$

has two connected components and $K = (G^\sigma)^\circ$ or $K = G^\sigma$. For example if $p = 1$

- $K = (G^\sigma)^\circ \implies G/K \cong \mathrm{SO}(n)/K \cong S^{n-1}$
- $K = G^\sigma \implies G/K = \mathbb{P}(\mathbb{R}^n)$

(4) The argument works similar for $\mathrm{U}(n)$.

We start the proof of Theorem II.17 with two lemmata that are good to emphasise.

Lemma II.19: Cartan decomposition

Let (G, K) be a Riemannian symmetric pair with an involutive automorphism σ, and let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Then

(1) $\mathfrak{k} = \{X \in \mathfrak{g} : \mathrm{d}_e\sigma X = X\}$, and

(2) if $\mathfrak{p} := \{X \in \mathfrak{g} : \mathrm{d}_e\sigma X = -X\}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Proof. (1) By definition of symmetric pair $\dim(G^\sigma)^\circ = \dim K = \dim(G^\sigma)$ so that, if $\mathfrak{k}$ is the Lie algebra of K,

$$\begin{aligned}
\mathfrak{k} &= \mathrm{Lie}(G^\sigma) \\
&= \{X \in \mathfrak{g} : \exp tX \in G^\sigma \text{ for all } t \in \mathbb{R}\} \\
&= \{X \in \mathfrak{g} : \sigma(\exp tX) = \exp tX \text{ for all } t \in \mathbb{R}\} \\
&\overset{(A.1)}{=} \{X \in \mathfrak{g} : \exp(\mathrm{d}_e\sigma(tX)) = \exp(tX), \text{ for all } t \in \mathbb{R}\} \\
&= \{X \in \mathfrak{g} : \mathrm{d}_e\sigma X = X\}.
\end{aligned}$$

In fact, since $(\mathrm{d}_e\sigma)^2 = Id$, $\mathrm{d}_e\sigma$ is diagonalizable with eigenvalues ± 1. If $X \in \mathfrak{g}$ is an eigenvector with eigenvalue -1, then $X \notin \mathrm{Lie}(G^\sigma)$ as it would otherwise contradict that the Lie group exponential is a local diffeomorphism (Proposition A.3)3)).**REVISE THE ARGUMENT**

(2) We write

$$X = \frac{1}{2}(X + \mathrm{d}_e\sigma X) + \frac{1}{2}(X - \mathrm{d}_e\sigma X)$$

and since $(\mathrm{d}_e\sigma)^2 = Id$, then $\frac{1}{2}(X + \mathrm{d}_e\sigma X) \in \mathfrak{k}$ and $\frac{1}{2}(X - \mathrm{d}_e\sigma X) \in \mathfrak{p}$. ■

Lemma II.20

Let (G, K) be a Riemannian symmetric pair with an involutive automorphism σ, and let $\mathfrak{p} := \{X \in \mathfrak{g} : \mathrm{d}_e\sigma X = -X\}$. Then $\mathfrak{p}$ is $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant.

Proof. Notice first that

$$\begin{aligned} \sigma \circ c_k(g) &= \sigma(kgk^{-1}) \\ &\overset{\sigma \in \mathrm{Aut}(G)}{=} \sigma(k)\sigma(g)\sigma(k)^{-1} \\ &\overset{K \subseteq G^\sigma}{=} k\sigma(g)k^{-1} \\ &= c_k \circ \sigma(g) \end{aligned}$$

and that by differentiation at the identity we get

$$(\mathrm{d}_e\sigma)(\mathrm{d}_e c_k) = \mathrm{d}_e(\sigma \circ c_k) = \mathrm{d}_e(c_k \circ \sigma) = (\mathrm{d}_e c_k)(\mathrm{d}_e\sigma).$$

As $\mathrm{Ad}_{\mathrm{G}}(k) = \mathrm{d}_e c_k$ we can rewrite this as

$$\mathrm{d}_e\sigma \circ \mathrm{Ad}_{\mathrm{G}}(k) = \mathrm{Ad}_{\mathrm{G}}(k) \circ \mathrm{d}_e\sigma.$$

Now if $X \in \mathfrak{p}$, we have $\mathrm{d}_e\sigma(X) = -X$ and we conclude that

$$\mathrm{d}_e\sigma(\mathrm{Ad}_{\mathrm{G}}(k)X) = \mathrm{Ad}_{\mathrm{G}}(k)(\mathrm{d}_e\sigma X) = \mathrm{Ad}_{\mathrm{G}}(k)(-X) = -\mathrm{Ad}_{\mathrm{G}}(k)(X).$$

that is,

$$\mathrm{Ad}_{\mathrm{G}}(k)X \in \mathfrak{p}$$ ■

Proof of Theorem II.17. First of all, the diagram

$$\begin{array}{ccc} G & \xrightarrow{c_k} & G \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/K & \xrightarrow{k} & G/K \end{array}$$

commutes and it follows by differentiation that

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_{\mathrm{G}}(k)} & \mathfrak{g} \\ {\scriptstyle \mathrm{d}_e\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_e\pi} \\ T_o\left({}^{G}\!\diagup_{K}\right) & \xrightarrow{\mathrm{d}_o k} & T_o\left({}^{G}\!\diagup_{K}\right) \end{array}$$

commutes as well, that is

$$\mathrm{d}_e\pi \circ \mathrm{Ad}_{\mathrm{G}}(k) = \mathrm{d}_o k \circ \mathrm{d}_e\pi.$$

Moreover, the differential $\mathrm{d}_e\pi\colon \mathfrak{g} \to T_o(G/K)$ is surjective (as π is surjective) and has kernel $\ker \mathrm{d}_e\pi = \mathfrak{k}$, so that we get the following commuting diagram

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\mathrm{Ad}_{\mathrm{G}}(k)} & \mathfrak{p} \\ {\scriptstyle \mathrm{d}_e\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_e\pi} \\ T_o\left({}^{G}\!\diagup_{K}\right) & \xrightarrow{\mathrm{d}_o k} & T_o\left({}^{G}\!\diagup_{K}\right) \end{array}$$

and

$$\mathfrak{p} \cong T_o\left({}^{G}\!\diagup_{K}\right)$$

not only as vector spaces, but also as K-spaces where the action of K on $\mathfrak{p}$ is via Ad_{G} on $T_o(G/K)$ is given by $\mathrm{d}_o k$.

Since $\mathrm{Ad}_{\mathrm{G}}(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, there exists a positive definite inner product B on $\mathfrak{p}$ and, actually, any positive definite inner product can be made $\mathrm{Ad}_G(K)$ invariant. In fact, if $B'\colon \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ is a positive definite inner product on $\mathfrak{p}$ and μ is the Haar measure on $\mathrm{Ad}_{\mathrm{G}}(K)$, then for $X, Y \in \mathfrak{p}$ the inner product

$$B(X,Y) \coloneqq \int_{\mathrm{Ad}_{\mathrm{G}}(K)} B'(k_* X, k_* Y)\, d\mu(k)$$

is obviously $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant and can be proven to be non-zero.

We set now

$$\begin{aligned} Q_o\colon T_o\left({}^{G}\!\diagup_{K}\right) \times T_o\left({}^{G}\!\diagup_{K}\right) &\longrightarrow \qquad\qquad \mathbb{R} \\ (X_o, Y_o) \qquad &\mapsto Q_o(X,Y) \coloneqq B(\mathrm{d}_e\pi^{-1} X_o, \mathrm{d}_e\pi^{-1} Y_o) \end{aligned}$$

which is now a K-invariant inner product on $T_o(G/K)$ and we extend it to $T_p(G/K)$ by pulling back $X_p, Y_p \in T_p(G/K)$, to $\mathrm{d}_o g^{-1} X_p, \mathrm{d}_o g^{-1} Y_p \in T_o(G/K)$, where $g(o) = p$ with $g \in G$. Thus

$$Q_p(X_p, X_p) \coloneqq Q_o(\mathrm{d}_o g^{-1} X_p, \mathrm{d}_o g^{-1} Y_p)\,, \tag{II.7}$$

Notice that this is well defined since Q_p is K-invariant. In fact, if $g(o) = p = h(o)$, then $h^{-1}g \in K$, so that

$$\begin{aligned} Q_o(\mathrm{d}_o g^{-1} X_p, \mathrm{d}_o g^{-1} Y_p) &= Q_o(\mathrm{d}_o(h^{-1}g)\mathrm{d}_o g^{-1} X_p, \mathrm{d}_o(h^{-1}g)\mathrm{d}_o g^{-1} Y_p) \\ &= Q_o(\mathrm{d}_o h^{-1} X_p, \mathrm{d}_o h^{-1} Y_p)\,. \end{aligned}$$

This gives a G-invariant Riemannian metric on G/K.

We need to define now the geodesic symmetries. We start with s_o. Once we'll have defined this, if $g(o) = p$ as above, then $s_o = g \circ s_o \circ g^{-1}$ will give the geodesic symmetry at any other point.

We define s_o as a map that satisfies the relation

$$s_o \circ \pi = \pi \circ \sigma \tag{II.8}$$

that is

$$s_o = \pi \circ \sigma \circ \pi^{-1}\,.$$

It is easy to see that s_o is well-defined. In fact, since $K \leq G^\sigma$, then

$$s_o(x) = \pi(\sigma(\pi^{-1}(x))) = \pi(\sigma(xk)) = \pi(\sigma(x)\sigma(k)) = \pi(\sigma(x)k) = \pi(\sigma(x))\,.$$

We see now that $s_o^2 = Id$. In fact, by applyingonce more s_o on the left of (II.8), we obtain

$$s_o \circ (s_o \circ \pi) = s_o \circ (\pi \circ \sigma) = (s_o \circ \pi) \circ \sigma = (\pi \circ \sigma) \circ \sigma = \pi \circ (\sigma)^2 = \pi\,,$$

so that $(s_o)^2 = Id$ as π is surjective.

Now we show that $\mathrm{d}_p s_p = -Id$. The commutativity of the diagram

$$\begin{CD} G @>\sigma>> G \\ @V\pi VV @VV\pi V \\ G/K @>s_o>> G/K \end{CD}$$

implies by differentiation that also

$$\begin{CD} \mathfrak{p} @>\mathrm{d}_e\sigma>> \mathfrak{p} \\ @V\mathrm{d}_e\pi VV @VV\mathrm{d}_e\pi V \\ T_o\left(G/K\right) @>\mathrm{d}_o s_o>> T_o\left(G/K\right) \end{CD}$$

commutes. Thus if $X \in \mathfrak{p}$,

$$\mathrm{d}_o s_o(\mathrm{d}_e\pi(X)) = \mathrm{d}_e\pi(\mathrm{d}_e\sigma(X)) = -\mathrm{d}_e\pi(X)\,,$$

that is $\mathrm{d}_o s_o = -Id$. Writing $p = (g)o$ and recalling that $s_{g(o)} = g s_o g^{-1}$ we see that also

$$\begin{aligned} \mathrm{d}_{g(o)} s_{g(o)} &= \mathrm{d}_{g(o)}(g s_o g^{-1}) \\ &= (\mathrm{d}_o g) \underbrace{(\mathrm{d}_o s_o)}_{=-Id} (\mathrm{d}_{g(o)} g^{-1}) \\ &= -(\mathrm{d}_o g)(\mathrm{d}_{g(o)} g^{-1}) \\ &= -Id \end{aligned}$$

We will use this to that s_p is an isometry, that is it preserves any G-invariant Riemannian metric Q

$$Q_p(X_p, Y_p) = Q_{s_o(p)}((\mathrm{d}_p s_o)X_p, (\mathrm{d}_p s_o)Y_p) \qquad \text{for all } p \in M, \forall X_p, Y_p \in T_p M$$

Before doing this, we have to gather some more information. Namely, we will see that the geodesic symmetry at o intertwines the isometry g and its image under σ. In other words, applying twice the formula (II.6) defining the geodesic symmetry s_o, we obtain that for $x \in G$

$$\begin{aligned} s_o \circ g(xK) &= s_o \circ \pi(gx) \\ &\overset{\text{(II.6)}}{=} \pi \circ \sigma(gx) \\ &= \sigma(gx)K \\ &= \sigma(g)\sigma(x)K \\ &= \sigma(g)(\pi \circ \sigma)(x) \\ &\overset{\text{(II.6)}}{=} \sigma(g)(s_o \circ \pi)(x) \\ &= \sigma(g) \circ s_o(xK)\,, \end{aligned}$$

that is

$$s_o \circ g = \sigma(g) \circ s_o\,. \tag{II.9}$$

Now

$$\begin{aligned} Q_{s_o(p)}((\mathrm{d}_p s_o)X_p, (\mathrm{d}_p s_o)Y_p) &= Q_{s_o(g(o))}((\mathrm{d}_{g(o)} s_o)(\mathrm{d}_o g)X_o, (\mathrm{d}_{g(o)} s_o)(\mathrm{d}_o g)Y_o) \\ &= Q_{s_o(g(o))}((\mathrm{d}_o(s_o \circ g)X_o, (\mathrm{d}_o(s_o \circ g)Y_o) \\ &\overset{\text{(II.9)}}{=} Q_{\sigma(g)(o)}((\mathrm{d}_o(\sigma(g) \circ s_o)X_o, (\mathrm{d}_o(\sigma(g) \circ s_o)Y_o) \\ &= Q_{\sigma(g)(o)}(\mathrm{d}_o\sigma(g) \underbrace{\mathrm{d}_o s_o X_o}_{=-X_o}, \mathrm{d}_o\sigma(g) \underbrace{\mathrm{d}_o s_o Y_o}_{=-Y_o}) \\ &= Q_{\sigma(g)(o)}(\mathrm{d}_o\sigma(g)X_o, \mathrm{d}_o\sigma(g)Y_o) \\ &\overset{\sigma(g)\in G}{=} Q_o(X_o, Y_o) \\ &= Q_o(\mathrm{d}_o g^{-1} X_p, \mathrm{d}_o g^{-1} Y_p) \\ &= Q_p(X_p, Y_p)\,. \end{aligned}$$

Hence s_o is an isometry. ■

Remark. Let M be a Riemannian symmetric space and (G, K) the associated Riemannian symmetric pair with regard to an involution $\sigma \in \mathrm{Aut}(G)$. We will prove that σ is unique.

Let σ_i, $i = 1, 2$, be two involutions of G such that

$$(G^{\sigma_i})^\circ \leq K \leq G^{\sigma_i} \qquad \text{for} \quad i = 1, 2.$$

Then

$$\pi \circ \sigma_1 = s_o \circ \pi = \pi \circ \sigma_2$$

and thus

$$\sigma_1(h)(o) = \sigma_2(h)(o) \qquad \text{for all } h \in G. \tag{II.10}$$

We still need to see that

$$\sigma_1(h)(p) = \sigma_2(h)(p) \qquad \forall h \in G, \forall p \in M.$$

Let thus $g \in G$ be such that $g(o) = p$ and let g' be such that $\sigma_1(g') = g$. Then

$$\begin{aligned}
\sigma_1(h)(p) &= \sigma_1(h)\sigma_1(g')(o) \\
&= \sigma_1(hg')(o) \\
&\overset{\text{(II.10)}}{=} \sigma_2(hg')(o) \\
&= \sigma_2(h)\sigma_2(g')(o) \\
&= \sigma_2(h)(p)
\end{aligned}$$

showing uniqueness.

The uniqueness of the involutive automorphism of a Riemannian symmetric pair allows us to give the following definition:

Definition: Cartan Involution

If (G, K) is a Riemannian symmetric pair with involution σ, the **Cartan involution** is defined as

$$\Theta := \mathrm{d}_e\sigma \colon \mathfrak{g} \to \mathfrak{g}.$$

The corresponding eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the **Cartan decomposition** of $\mathfrak{g}$ with respect to Θ.

Remark. We saw in Lemma II.19 that such a decomposition exists and now we also know that it is unique.

Proposition II.21

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution Θ. Then

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,.$$

In particular $\mathfrak{k} \subset \mathfrak{g}$ is a Lie subalgebra, while $\mathfrak{p} \subset \mathfrak{g}$ is only a subvector space.

Proof. Let $X, Y \in \mathfrak{g}$ be eigenvectors with eigenvalues $\lambda, \mu \in \{\pm 1\}$ respectively. Then

$$\Theta[X, Y] = [\Theta X, \Theta Y] = [\lambda X, \mu Y] = \lambda\mu[X, Y]\,,$$

that is $[X, Y]$ belongs to the eigenspace of Θ with eigenvalue $\lambda\mu$. ∎

II.6 Exponential Maps and Geodesics

Let (G, K) be a Riemannian symmetric pair associated to a a Riemannian symmetric space M with base point $o \in M$. By the last Remark, there is a unique involution σ and hence the Cartan decomposition of $\mathfrak{g}$ is unique. Let $\pi\colon G \to M$ be the projection map $g \mapsto g(o)$, let $\exp\colon \mathfrak{g} \to G$ be the Lie group exponential map and $\operatorname{Exp}_o\colon T_oM \to M$ the Riemannian exponential map.

The following theorem gives the relation between the two exponential maps, namely:

Theorem II.22

The following diagram

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\mathrm{d}_e\pi} & T_oM \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \operatorname{Exp}_o} \\ G & \xrightarrow{\pi} & M \end{array}$$

commutes, that is $\pi(\exp(X)) = \operatorname{Exp}_p(\mathrm{d}_e\pi(X))$ for any $X \in \mathfrak{p}$.
In particular, if $X \in \mathfrak{p}$, then

$$t \mapsto (\exp(tX))(o) \in M$$

is the geodesic through $o \in M$ at $t = 0$ with tangent vector $\mathrm{d}_e\pi(X) \in T_o(M)$.

Proof. If $X \in \mathfrak{p}$, let $\gamma(t) := \operatorname{Exp}_o(t\mathrm{d}_e\pi(X))$ the geodesic in M through o at $t = 0$ with tangent vector $\mathrm{d}_e\pi(X) \in T_oM$. Let $\mathcal{T}_{\gamma,t} = s_{\gamma(t/2)} \circ s_{\gamma(0)} = s_{\gamma(t/2)} \circ s_o$ be the transvection along γ. Since $\mathcal{T}_{\gamma,t}$ is a one-parameter subgroup in G, there exists $Y \in \mathfrak{g}$ such that $\mathcal{T}_{\gamma,t} = \exp(tY) \in G$. We claim that actually $Y \in \mathfrak{p}$, that is that $d_e\sigma(Y) = -Y$. In fact, using that $\sigma(g) = s_{\gamma(0)} g s_{\gamma(0)}$ we have

$$\begin{aligned}
\exp(t\mathrm{d}_e\sigma(Y)) &= \exp(\mathrm{d}_e\sigma(tY)) \\
&= \sigma(\exp(tY)) \\
&= s_{\gamma(0)} \exp(tY) s_{\gamma(0)} \\
&= s_{\gamma(0)} \mathcal{T}_{\gamma,t} s_{\gamma(0)} \\
&= s_{\gamma(0)} s_{\gamma(t/2)} \underbrace{s_{\gamma(0)} s_{\gamma(0)}}_{=Id} \\
&= s_{\gamma(0)}^{-1} s_{\gamma(t/2)}^{-1} \\
&= \left(s_{\gamma(t/2)} s_{\gamma(0)}\right)^{-1} \\
&= (\mathcal{T}_{\gamma,t})^{-1} \\
&= \mathcal{T}_{\gamma,-t} \\
&= \exp(-tY).
\end{aligned}$$

Observe that

$$\begin{aligned}
\pi(\exp(tY)) &= \pi(\mathcal{T}_{\gamma,t}) \\
&= \pi\left(s_{\gamma(t/2)} \circ s_{\gamma(0)}\right) \\
&= s_{\gamma(t/2)} \circ s_{\gamma(0)}(o) \\
&= s_{\gamma(t/2)} \gamma(0) \\
&= \gamma(t) \\
&= \mathrm{Exp}_o(t\mathrm{d}_e\pi(X)),
\end{aligned}$$

so it is enough to show that $X = Y$.

Since the tangent vector at $t = 0$ to the geodesic $t \mapsto \mathrm{Exp}_o(t\mathrm{d}_e\pi(X))$ is $\mathrm{d}_e\pi(X)$, it will be enough to show that the tangent vector at $t = 0$ to $t \mapsto \pi(\exp(tY))$ is $\mathrm{d}_e\pi(Y)$. Since $\mathrm{d}_e\pi$ is an isomorphism on $\mathfrak{p}$, this would conclude that $X = Y$. To find the tangent vector at $t = 0$ to $t \mapsto \pi(\exp(tY))$, we evaluate the derivative at $t = 0$ and obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tY)) = \mathrm{d}_e\pi \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tY) = \mathrm{d}_e\pi(Y)$$

as we wanted. ■

The above theorem shows, in particular, that the Riemannian exponential map $\mathrm{Exp}\colon TM \to M$ does not depend on its Riemannian metric and gives a formula for the geodesics in M.

We are now interested in finding a formula for the derivative of the Riemannian exponential map at a point $X \in \mathfrak{p}$, a formula that we will use both in computing the curvature tensor in § II.11 and in characterizing the totally geodesic submanifolds of a Riemannian symmetric space in § II.7.

Theorem II.23

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp\colon \mathfrak{g} \to G$ the Lie group exponential map. By identifying $T_X\mathfrak{g} \cong \mathfrak{g}$, we have that

$$\mathrm{d}_X \exp\colon T_X\mathfrak{g} \cong \mathfrak{g} \to T_{\exp(X)}G$$

is given by

$$\mathrm{d}_X \exp = \mathrm{d}_e L_{\exp X} \circ \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{\mathfrak{g}}^n X)}{(n+1)!} \tag{II.11}$$

Let $M = G/K$ be a symmetric space, $o \in M$ a base point with $K = \mathrm{Stab}_G(o)$ and $\pi\colon G \to G/K$. Recall that $\mathrm{d}_e\pi\colon \mathfrak{p} \to T_o(G/K)$ is an isomorphism and we can define $\mathrm{Exp} \circ \mathrm{d}_e\pi\colon \mathfrak{p} \to T_o(G/K) \to G/K$. Then we have:

Corollary II.24

The differential

$$\mathrm{d}_X(\mathrm{Exp} \circ \mathrm{d}_e\pi)\colon T_X\mathfrak{p} \cong \mathfrak{p} \to T_{(\mathrm{Exp}(X)\circ \mathrm{d}_e(X))(o)}M$$

of the Riemannian exponential map

$$\mathrm{Exp}_o \circ \mathrm{d}_e\pi\colon \mathfrak{p} \to G/K$$

is given by

$$\mathrm{d}_X(\mathrm{Exp}_o \circ \mathrm{d}_e\pi) = \mathrm{d}_o L_{\exp X} \circ \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!}\,, \tag{II.12}$$

where $T_X = (\mathrm{ad}_{\mathfrak{g}} X)^2$ for $X \in \mathfrak{p}$.

Proof. We recall that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & G/K \\ \Big\downarrow{\scriptstyle L_g} & & \Big\downarrow{\scriptstyle L_g} \\ G & \xrightarrow{\ \pi\ } & G/K \end{array}$$

commutes, so that

$$\pi \circ L_{\exp X} = L_{\exp X} \circ \pi\,. \tag{II.13}$$

In Theorem II.22 we have proven that for any $X \in \mathfrak{p}$, $\pi \circ \exp(X)|_{\mathfrak{p}} = \mathrm{Exp}_e \circ \mathrm{d}_e\pi X|_{\mathfrak{p}}$, so that, if we set $\mathcal{L}(X) \coloneqq \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}_{\mathfrak{g}} X)^n}{(n+1)!}$,

$$\begin{aligned}
\mathrm{d}_X(\mathrm{Exp}_0 \circ \mathrm{d}_e\pi X|_{\mathfrak{p}}) =& \mathrm{d}_X(\pi \circ \exp(X)|_{\mathfrak{p}}) \\
=&(\mathrm{d}_{\exp X}\pi) \circ \mathrm{d}_X(\exp|_{\mathfrak{p}}) \\
=&(\mathrm{d}_{\exp X}\pi) \circ \mathrm{d}_X(\exp)|_{\mathfrak{p}} \\
=&(\mathrm{d}_{\exp X}\pi) \circ \mathrm{d}_e L_{\exp X} \circ \mathcal{L}(X) \\
=&\mathrm{d}_e(\pi \circ L_{\exp X}) \circ \mathcal{L}(X)|_{\mathfrak{p}} \\
=&\mathrm{d}_e(L_{\exp X} \circ \pi) \circ \mathcal{L}(X)|_{\mathfrak{p}} \\
=&(\mathrm{d}_o L_{\exp X}) \circ (\mathrm{d}_e\pi) \circ \mathcal{L}(X)|_{\mathfrak{p}}
\end{aligned}$$

where we used in the fourth equality Theorem II.23 and in the sixth one (II.13).

Now observe that, because of Proposition II.21, if $Y \in \mathfrak{p}$,

$$\operatorname{ad}_{\mathfrak{g}}(X)^n(Y) \in \begin{cases} \mathfrak{k} & \text{if } n \text{ is odd} \\ \mathfrak{p} & \text{if } n \text{ is even,} \end{cases}$$

so that

$$\mathrm{d}_e\pi \circ \operatorname{ad}_{\mathfrak{g}}(X)^n(Y) \begin{cases} = 0 & \text{if } n \text{ is odd} \\ = \operatorname{ad}_{\mathfrak{g}}(X)^n(Y) & \text{if } n \text{ is even.} \end{cases}$$

Thus

$$\mathrm{d}_e\pi \circ \mathcal{L}(X)|_{\mathfrak{p}} = \mathrm{d}_e \circ \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}_{\mathfrak{g}} X)^n}{(n+1)!}\Bigg|_{\mathfrak{p}} = \sum_{n=0}^{\infty} \frac{(\operatorname{ad}_{\mathfrak{g}} X)^{2n}}{(2n+1)!}\,, \tag{II.14}$$

which completes the proof. ■

II.7 Totally Geodesic Submanifolds

Definition: Totally Geodesic Submanifolds

Let $N \subset M$ be a submanifold of a Riemannian manifold (M, g). We say that N is **geodesic at** $p \in N$ if for every $v \in T_pN$ the M-geodesic through p with tangent vector v is all contained in N.
We say that N is **totally geodesic** if it is geodesic at every point.

Remark. If (M, g) is a Riemannian manifold and $N \subset M$ a submanifold, then $g|_N$ is a Riemannian metric on N. A priori, if $p, q \in N$, then

$$d_M(p, q) \leq d_N(p, q)$$

where d_M, d_N are the distances induced by the metrics g and $g|_N$ respectively.

Fact. Assume $N \subset M$ totally geodesic. Then

(1) the inclusion $(N, d_N) \hookrightarrow (M, d_M)$ is locally distance preserving and

(2) every N-geodesic is an M-geodesic and every M-geodesic contained in N is an N-geodesic.

Example. Totally geodesic submanifolds are not very common.

(1) In $\mathbb{R}^n$ all linear subspaces and their translates are totally geodesic. $S^2 \subset \mathbb{R}^3$ however is not.

(2) In S^n the totally geodesic subspaces are the intersection of S^n with a linear subspace of $\mathbb{R}^{n+1}$.

(3) (Cartan) Let M be a Riemannian manifold such that for any $p \in M$ and every 2-dimensional plane $P \subset T_pM$, there exists a totally geodesic submanifold through p which is tangent to P. Then M has constant curvature.

Theorem II.25

Let (M, g) be a Riemannian manifold and $N \subset M$ a connected submanifold. Then N is totally geodesic if and only if the parallel transport with respect to g along curves in N preserves the tangent spaces (i.e. parallel transport preserves $\{T_pN : p \in N\}$).

Example. (Being totally geodesic is a local property)

$$\mathbb{T}^n = \mathbb{E}^n / \mathbb{Z}^n \qquad \pi\colon \mathbb{E}^n \to \mathbb{T}^n$$

If $P \subset \mathbb{E}^n$ is a k-dimensional subspace, $k < n$, then $\pi(P)$ is a totally geodesic submanifold of $\mathbb{T}^n$. However, P can be chosen is such a way that $\pi(P)$ is dense in $\mathbb{T}^n$ (e.g. $n = 2$ and P the irrational line).

Definition: Lie Triple System

A subspace $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is a **Lie triple system** if $[[X, Y], Z] \in \mathfrak{n}$ for all $X, Y, Z \in \mathfrak{n}$.

Example. $\mathfrak{p} \subset \mathfrak{g}$ since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$.

Lie triple systems correspond to totally geodesic submanifolds in the following sense:

Theorem II.26

Let $M = G/K$ be a Riemannian symmetric space with $o \in M$ a base point, $K = \mathrm{Stab}_o(G)$ where $G = \mathrm{Iso}(M)^\circ$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

(1) If $\mathfrak{n} \subset \mathfrak{p}$ is a Lie triple system, then $N := (\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(\mathfrak{n}) \subset M$ is a totally geodesic submanifold through $o \in M$ and such that $T_oN = \mathrm{d}_e\pi(\mathfrak{n})$.

(2) If $N \subset M$ is a totally geodesic submanifold through o, then $\mathfrak{n} := (\mathrm{d}_e\pi)^{-1}(T_oN)$ is a Lie triple system.

Remark. If $N \subset M$ is a totally geodesic submanifold, let $p \in N$ and $g \in G$ be such that $g(o) = p$. Then $L_g^{-1}(N)$ is a totally geodesic submanifold through o to which one can apply the theorem.

Lemma II.27

If $\mathfrak{n} \subset \mathfrak{g}$ is a Lie triple system, then

- *$[\mathfrak{n}, \mathfrak{n}]$ is a subalgebra and*
- *$\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$ is a subalgebra.*

Proof. If $X, Y, Z, W \in \mathfrak{n}$, then, the Jacobi identity applied to $[X, Y]$, Z and W reads

$$0 = [[X, Y], [Z, W]] + [[Y, [Z, W]], X] + [[[Z, W], X], Y],$$

where $[Y, [Z, W]], [[Z, W], X] \in \mathfrak{n}$. Hence

$$[[X, Y], [Z, W]] = -[[Y, [Z, W]], X] - [[[Z, W], X], Y] \in [\mathfrak{n}, \mathfrak{n}].$$

It follows that

$$\begin{aligned} [\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}], \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]] &\subset [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] + [[\mathfrak{n}, \mathfrak{n}], [\mathfrak{n}, \mathfrak{n}]] \\ &\subset \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]. \end{aligned}$$

■

Proof of Theorem II.26. (1) Let $\mathfrak{n} \subset \mathfrak{p}$ be a Lie triple system. By the lemma, $\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{g}$ is a subalgebra and $\mathfrak{g} = \mathrm{Lie}(G)$. Let now $G' < G$ be the connected Lie subgroup such that

$$\mathrm{Lie}(G') = \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}] := \mathfrak{g}'.$$

Let then

$$\begin{aligned} G' &\to M \\ g' &\mapsto g'(o) \end{aligned}$$

and let $K' = \mathrm{Stab}_{G'}(p_0)$. Then $K' < G'$ is closed since the inclusion $G' \hookrightarrow G$ is continuous. Thus we can give $M' = G' \cdot o$ the topology and differential structure of G'/K'. It follows that M' is a submanifold of M and $o \in M' \subset M$ is a base point.

We claim now that $T_oM' = \mathrm{d}_e\pi(\mathfrak{n})$. Since $M' = G'/K'$ and $\mathrm{Lie}(G') = \mathfrak{g}' = \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$ it is enough to see that $\mathrm{Lie}(K') = [\mathfrak{n}, \mathfrak{n}]$. In fact, $K' = K \cap G'$ and thus

$$\mathrm{Lie}(K') = \mathfrak{k} \cap (\mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]) = [\mathfrak{n}, \mathfrak{n}]$$

since $\mathfrak{n} \subset \mathfrak{p}$ (and hence $\mathfrak{k} \cap \mathfrak{n} = (0)$) and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{k}$ (since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$). So given a Lie triple system $\mathfrak{n}$, we found a submanifold $M' \subset M$ whose tangent space is the Lie triple system. We are left to show that M' is totally geodesic.

Let $X \in \mathfrak{n}$, $v := (\mathrm{d}_e\pi)(X) \in T_oM'$. The M-geodesic through o with tangent vector v is

$$t \mapsto \exp(tX)_*(o) = \mathrm{Exp}_o(tv).$$

But $\forall t \in \mathbb{R}$, $tX \in \mathfrak{n}$ and thus $\exp(tX) \in G'$ which implies $\exp(tX)_* \cdot o \in M'$ and hence that M' is totally geodesic.

(2) We want to show that if $N \subset M$ is totally geodesic, then $\mathfrak{n} := (\mathrm{d}_e\pi)^{-1}T_oN$ is a Lie triple system.

<u>Claim:</u> If $X, Y \in \mathfrak{n}$, then $T_X(Y) \in \mathfrak{n}$.

Using this we want to show that if $X, Y, Z \in \mathfrak{n}$, then $[[X, Y], Z] \in \mathfrak{n}$. In particular we observe

$$\begin{aligned} T_{Y+Z}(X) &= \mathrm{ad}_{\mathfrak{g}}(Y+Z)(\mathrm{ad}_{\mathfrak{g}}(Y+Z)(X)) \\ &= [Y+Z, [Y+Z, X]] \\ &= [Y+Z, [Y, X] + [Z, X]] \\ &= [Y, [Y, X]] + [Y, [Z, X]] + [Z, [Y, X]] + [Z, [Z, X]] \\ &= T_Y(X) + T_Z(X) + [Y, [Z, X]] + [Z, [Y, X]], \end{aligned}$$

so that

$$[Y, [Z, X]] + [Z, [Y, X]] = T_{Y+Z}(X) - T_Y(X) - T_Z(X) \in \mathfrak{n}.$$

By the Jacobi identity

$$\begin{aligned}\mathfrak{n} \ni [Y,[Z,X]] + [Z,[Y,X]] =& [Y,[Z,X]] + [[Z,Y],X] + [Y,[Z,X]] \\ =& 2[Y,[Z,X]] + [[Z,Y,X]] \\ =& 2[Y,[Z,X]] + [X,[Y,Z]]\end{aligned} \tag{II.15}$$

and, exchanging the roles of X and Y,

$$2[X,[Z,Y]] + [Y,[X,Z]] \in \mathfrak{n}\,. \tag{II.16}$$

Hence it follows from (II.15) and (II.16), and using twice the Jacobi identity, that

$$\begin{aligned}\mathfrak{n} \ni& 2[Y,[Z,X]] + [X,[Y,Z]] - (2[X,[Z,Y]] + [Y,[X,Z]]) \\ =& 2[X,[Z,Y]] + 2[Z,[Y,X]] + [X,[Y,Z]] - (2[X,[Z,Y]] + [Y,[X,Z]]) \\ =& 3[Y,[Z,X]] + 3[X,[Y,Z]] \\ =& 3[Z,[Y,X]]\,,\end{aligned}$$

that is $\mathfrak{n}$ is a Lie triple system.

Proof of Claim: Take $X \in \mathfrak{n} = (\mathrm{d}_e\pi)^{-1}T_oN$. Then we clearly have $\mathrm{d}_e\pi(X) \in T_oN$ and

$$(\exp(tX))(o) = (\mathrm{Exp}_o\, \mathrm{d}_e\pi(tX))$$

is an M-geodesic through o such that the tangent vector at o is $\mathrm{d}_e\pi(X) \in T_oN$. As N is totally geodesic,

$$t \mapsto \mathrm{Exp}_o \circ \mathrm{d}_e\pi(X) \in N \qquad \forall t \in \mathbb{R}.$$

Now

$$\mathfrak{n} \xrightarrow{\mathrm{d}_e\pi} T_oN \xrightarrow{\mathrm{Exp}_o} N$$

and thus

$$\mathrm{d}_{tX}(\mathrm{Exp}_o \circ \mathrm{d}_e\pi)\colon T_{tX}\mathfrak{n} \cong \mathfrak{n} \to T_{(\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(tX)}N$$

It follows from Corollary II.24 that, since $\mathfrak{n} \subset \mathfrak{p}$, for all $Y \in \mathfrak{n}$,

$$\mathrm{d}_{tX}(\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(Y) = \mathrm{d}_oL_{\exp tX} \circ \mathrm{d}_e\pi\left(\sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!}\right).$$

Applying the inverse of $\mathrm{d}_oL_{\exp(tX)}$ to both sides we get

$$(\mathrm{d}_oL_{\exp(tX)})^{-1} \underbrace{\mathrm{d}_{tX}(\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(Y)}_{\in T_{(\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(tX)}} = \mathrm{d}_e\pi\left(\sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!}\right)$$

and we claim that

$$\mathrm{d}_e\pi\left(\sum_{n=0}^{\infty}\frac{(T_{tX})^n(Y)}{(2n+1)!}\right)\in T_oN$$

By Theorem II.22 the curve $t\mapsto \exp(tX)(o)$ is a geodesic through o and $\mathcal{T}_{\gamma,t}=\exp(tX)$ is the transvection whose derivative $d\mathcal{T}_{\gamma,t}\colon T_{\gamma(0)}\to T_{\gamma(t)}$ realizes the parallel transport along γ (Proposition II.15). Therefore, $\mathrm{d}_o(L_{\exp(tX)})^{-1}$ is the parallel transport along $t\mapsto\exp(tX)$. Since N is totally geodesic, this geodesic is completely contained in N and parallel transport preserves $\{T_pN : p\in N\}$:

$$\sum_{n=0}^{\infty}\frac{T_{tX}^n(Y)}{(2n+1)!}\in\mathfrak{n}$$

We write

$$\begin{aligned}\phi(t) := &\sum_{n=0}^{\infty}\frac{T_{tX}^n(Y)}{(2n+1)!}\\ =&\sum_{n=0}^{\infty}\frac{\mathrm{ad}_{\mathfrak{g}}(tX)^{2n}(Y)}{(2n+1)!}\\ =&\,t^2\frac{\mathrm{ad}_{\mathfrak{g}}(X)^2(Y)}{3!}+t^4(...)\end{aligned}$$

and

$$\phi''(t)\Big|_{t=0}=\frac{1}{3}(\mathrm{ad}_{\mathfrak{g}}(X))^2(Y)=\frac{1}{3}T_X(Y)\in\mathfrak{n}$$

which concludes the proof.

■

Remark. Let (G,K) be a Riemannian symmetric pair with involution σ and let $\mathfrak{n}\subset\mathfrak{g}$ be a Lie triple system with associated totally geodesic submanifold $N=\mathrm{Exp}(\mathfrak{n})$ through o. Then we saw that $N'=G'/K.$, where G' is such that $\mathrm{Lie}(G')=\mathfrak{n}+[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{g}$ and $K'=K\cap G'$ is such that $\mathrm{Lie}(K')=[\mathfrak{n},\mathfrak{n}]$. We look for the Riemannian symmetric pair associated to N. If $\Theta\coloneqq\mathrm{d}_e\sigma$ is the Cartan involution, then

$$\Theta X=-X\qquad\forall X\in\mathfrak{n}\subset\mathfrak{p}$$

and moreover

$$\Theta([\mathfrak{n},\mathfrak{n}])=[\Theta(\mathfrak{n}),\Theta(\mathfrak{n})]\subset[\mathfrak{n},\mathfrak{n}]\,.$$

Thus

$$\Theta(\mathfrak{g}')=\mathfrak{g}'\qquad\text{and}\qquad\sigma(G')=G'\,,$$

since G' is connected.

Let now $\sigma' := \sigma|_{G'}$ be an involution of G'. We want to show that

$$((G')^{\sigma'})^\circ \leq K' \leq (G')^{\sigma'}.$$

It so, then (G', K') is a Riemannian symmetric pair associated to N, which is therefore a Riemannian symmetric space $N \cong G'/K'$.

Note now that

$$\begin{aligned} K' &= K \cap G' \leq (G^\sigma) \cap G' \leq (G')^{\sigma'} \\ K' &= K \cap G' \geq (G^\sigma)^\circ \cap G' \end{aligned}$$

but $(G^\sigma)^\circ \cap G'$ is an open subgroup of G' and thus

$$(G^\sigma)^\circ \cap G' \geq ((G')^{\sigma'})^\circ .$$

II.8 Example: Riemannian Symmetric Pair $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(n,\mathbb{R}))$

Let us consider

$$G = \mathrm{SL}(n,\mathbb{R}) = \{g \in M_{n\times n}(\mathbb{R}) : \det g = 1\} .$$

We consider the involutive automorphism $\sigma\colon G \to G$, defined by $g \mapsto (g^t)^{-1}$. Then

$$G^\sigma = \{g \in G : (g^t)^{-1} = g\} = \{g \in G : g^t g = e\} = \mathrm{SO}(n) =: K .$$

$\mathrm{SO}(n,\mathbb{R})$ is compact, thus $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(n,\mathbb{R}))$ is a Riemannian symmetric pair.

Since $\mathrm{d}_e \det = \mathrm{tr}$, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(n,\mathbb{R})$ consists of all $(n\times n)$-matrices with trace 0 and entries in $\mathbb{R}$ and the exponential map $\exp\colon \mathfrak{sl}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is the matrix exponential

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

It follows that

$$\exp\bigl(X^t\bigr) = (\exp(X))^t$$

and thus also that

$$\sigma(\exp(tX)) = \exp\bigl(-tX^t\bigr).$$

It is then immediate that the Cartan involution $\Theta\colon \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{s},\mathbb{R})$ is given by

$$\Theta(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp\bigl(-tX^t\bigr) = -X^t$$

By noting that

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{sl}(n,\mathbb{R}) : \Theta X = X\} \\ &= \left\{X \in \mathfrak{sl}(n,\mathbb{R}) : -X^t = X\right\} \\ \mathfrak{p} &= \left\{X \in \mathfrak{sl}(n,\mathbb{R}) : X^t = X\right\}\end{aligned}$$

the Cartan decomposition is given

$$X = \underbrace{\frac{1}{2}(X - X^t)}_{\in\mathfrak{k}} + \underbrace{\frac{1}{2}(X + X^t)}_{\in\mathfrak{p}},$$

that is, the decomposition of X into its antisymmetric and symmetric part.

We then want an $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant inner product on $\mathfrak{p}$. For this we recall that $\mathrm{Ad}_{\mathrm{G}}\colon G \to \mathrm{GL}(\mathfrak{g})$ is conjugation since $G < \mathrm{GL}(n,\mathbb{R})$

$$\mathrm{Ad}_{\mathrm{G}}(g)(X) = gXg^{-1}.$$

Since $M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^2$, the inner product

$$\begin{aligned}M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) &\longrightarrow \quad \mathbb{R} \\ (A,B) \qquad &\mapsto \mathrm{Tr}\big(A^t B\big)\end{aligned}$$

corresponds to the standard inner product on $\mathbb{R}^{n^2}$ and hence is clearly $\mathrm{Ad}_{\mathrm{G}}(O(n,\mathbb{R}))$-invariant. On $\mathfrak{p}$ this actually reduces to

$$(A,B) \mapsto \mathrm{Tr}(AB).$$

Consider the model

$$\mathcal{P}^1(n) = \left\{S \in M_{n\times n}(\mathbb{R}) : S^t = S, \det S = 1, S \gg 0\right\}$$

for $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{R})$ where $\mathrm{SL}(n,\mathbb{R})$ acts on $\mathcal{P}^1(n)$ via

$$g \cdot S \coloneqq gS^t g.$$

Notation. Take $Id \in \mathcal{P}^1(n)$ as base point. We proved in Theorem II.22 that if $X \in \mathfrak{p}$, then

$$(\mathrm{Exp}_{Id} \circ \mathrm{d}_e\pi)(X) = (\exp X) \cdot Id$$

and we thus write

$$\mathrm{Exp} \coloneqq \mathrm{Exp}_{Id} \circ \mathrm{d}_e\pi$$

that is,

$$\mathfrak{p} \xrightarrow{\mathrm{d}_e\pi} T_{\mathbb{1}}\mathcal{P}'^1 n) \xrightarrow{\mathrm{Exp}_{\mathbb{1}}} \mathcal{P}^1(n), \qquad \mathfrak{p} \xrightarrow{\mathrm{Exp}} \mathcal{P}^1(n)$$

Fact. $\operatorname{Exp}\colon \mathfrak{p} \to \mathcal{P}^1(n)$ is a diffeomorphism and if we consider $\operatorname{Exp}(0), \operatorname{Exp}(X) \in \mathcal{P}^1(n)$, then there exists a unique geodesic between those two points ($t \mapsto \operatorname{Exp}(tX)$) and this geodesic is length minimizing

$$d(\operatorname{Exp}(X), \operatorname{Exp}(0)) = \|X\|.$$

Let now $X \in \mathfrak{p}$ and note that

$$\begin{aligned} \operatorname{Exp}(X) &= (\exp X)_* \mathbb{1} \\ &= \exp(X) \mathbb{1} \exp(tX) \\ &= \exp(2X) \end{aligned}$$

and

$$\begin{aligned} \exp(-X)_* \operatorname{Exp}(X) &= \exp(-X) \operatorname{Exp}(X) \exp(-X) \\ &= \exp(-X) \exp(2X) \exp(-X) \\ &= \mathbb{1} \in \mathcal{P}'(n). \end{aligned}$$

Let also

$$\mathfrak{a} = \left\{ \operatorname{diag}(x_1, ..., x_n) : \sum x_i = 0 \right\}$$

such that $\mathfrak{a} \subset \mathfrak{p}$ since $\mathfrak{a}^t = \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. It follows that $\mathfrak{a}$ is a Lie triple system and thus also that

$$F = \operatorname{Exp}(\mathfrak{a}) = \left\{ \operatorname{diag}(x_1, ..., x_n) : \prod x_i = 1 \right\}$$

is a totally geodesic submanifold.

We compute the distance in F. Take $X_1, X_2 \in \mathfrak{a}$ and $\operatorname{Exp}(X_1), \operatorname{Exp}(X_2) \in F$. Then

$$\begin{aligned} d(\operatorname{Exp}(X_1), \operatorname{Exp}(X_2)) &= d(\exp(-X_2)_* \operatorname{Exp}(X_1), \mathbb{1}) \\ &= d(\exp(-X_2) \exp(2X_1) \exp(-X_2), \mathbb{1}) \\ &\overset{[X_1, X_2]=0}{=} d(\exp(2X_1 - 2X_2), \mathbb{1}) \\ &\overset{\text{Fact}}{=} \|X_1 - X_2\| \end{aligned}$$

We have thus shown that $\operatorname{Exp}\colon \mathfrak{a} \to F$ is an isometry. We call F a flat and note that $\dim(F) = n - 1$. More generally we will see that $\mathfrak{sl}(n, \mathbb{R})$ contains a maximal Abelian subalgebra that is diagonalizable ($= \mathfrak{a}$) and we will call the dimension the rank.

II.9 Decomposition of Symmetric Spaces

II.9.1 orthogonal Symmetric Lie Algebras

We have seen that a globally symmetric space M together with the choice of a base point $o \in M$ gives rise to a pair $(\mathfrak{g}, \Theta)$, where $\mathfrak{g}$ is the Lie algebra of (the connected component of) the group of isometries of M and Θ is the Cartan involution, that is the differential $\Theta = \mathrm{d}_e\sigma$ of the involutive automorphism σ of G induced by the geodesic symmetry at o.

Recall. The **Killing Form** of a Lie algebra $\mathfrak{g}$ is a bilinear symmetric form

$$\begin{aligned} B_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{K} = \text{field of definition of } \mathfrak{g} \\ (X, Y) &\mapsto \mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(Y)) \end{aligned}$$

Recall also the following properties of the Killing form:

(1) If $\alpha \in \mathrm{Aut}(\mathfrak{g})$, then

$$B_{\mathfrak{g}}(\alpha(X), \alpha(Y)) = B_{\mathfrak{g}}(X, Y) \qquad \forall X, Y \in \mathfrak{g}$$

(2) If $D \in \mathrm{Der}(\mathfrak{g})$ is a derivation, that is, D satisfies

$$D[X, Y] = [DX, Y] + [X, DY]$$

then we have that

$$B_{\mathfrak{g}}(DX, Y) + B_{\mathfrak{g}}(X, DY) = 0.$$

In particular, if $Z \in \mathfrak{g}$, then $\mathrm{ad}_{\mathfrak{g}}(Z) \in \mathrm{Der}(\mathfrak{g})$ and hence

$$B_{\mathfrak{g}}(\mathrm{ad}_{\mathfrak{g}}(Z)(X), Y) + B_{\mathfrak{g}}(X, \mathrm{ad}_{\mathfrak{g}}(Z)(Y)) = 0$$

Properties of $(\mathfrak{g}, \Theta)$ related to a Riemannian symmetric pair (G, K).

(1) Θ is an involution of $\mathfrak{g}$ and $\mathrm{Lie}(K) = \mathfrak{k}$ is the eigenspace with eigenvalue 1.

(2) $\mathrm{ad}_{\mathfrak{g}} = \mathrm{d}_e\mathrm{Ad}_{\mathrm{G}}$ and since K is compact, $\mathrm{Ad}_{\mathrm{G}}(K) < GL(\mathfrak{g})$ is a compact subgroup. Moreover, $\mathrm{Lie}(\mathrm{Ad}_{\mathrm{G}}(K)) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$.

Definition: Compactly Embedded

Let $\mathfrak{g}$ be a Lie algebra. We say that a subalgebra $\mathfrak{u} \subset \mathfrak{g}$ is **compactly embedded** if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of a compact subgroup $U < GL(\mathfrak{g})$.

Remark. We would like to say that $U = \mathrm{Ad_G}(K)$, $K < G$ compact and $K \cong U$ but K might have a center

$$U = {}^{\mathrm{Ad_G}(K)}\!\diagup_{Z(G) \cap K}$$

Fact. Any such group U is a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Proof: By naturality of the adjoint representation we have for all $t \in \mathbb{R}, X \in \mathfrak{g}$

$$\mathrm{Ad_G}(\exp(tX)) = \exp(\mathrm{ad}_{\mathfrak{g}}(tX)).$$

But $\mathrm{Ad_G}(g) = \mathrm{d}_e c_g$ which implies that $\mathrm{Ad_G}(g) \in \mathrm{Aut}(\mathfrak{g})$ for every $g \in G$. It follows that

$$\mathrm{Lie}(U) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subset \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$$

and thus that $U^\circ < \mathrm{Aut}(\mathfrak{g})$. Since U is connected, this implies that $U \subset \mathrm{Aut}(\mathfrak{g})$ which concludes the proof.

Definition: (Effective) orthogonal Symmetric Lie Algebra

(1) An **orthogonal symmetric Lie algebra** (OSLA) is a pair $(\mathfrak{g}, \Theta)$, where $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ and $\Theta \in \mathrm{Aut}(\mathfrak{g})$ is an involutive automorphism of $\mathfrak{g}$ such that its set of fixed points $\mathfrak{u} := \{X \in \mathfrak{g} : \Theta X = X\}$ is a compactly embedded subalgebra of $\mathfrak{g}$.

(2) The orthogonal symmetric Lie agebra $(\mathfrak{g}, \Theta)$ is **effective** if $\mathfrak{g} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z} \subset \mathfrak{g}$ is the center of $\mathfrak{g}$.

Remark. We note that since Θ is an involution ($\Theta^2 = Id$) it can only have the eigenvalues ± 1. We write

$$\mathfrak{u} = \{X \in \mathfrak{g} : \Theta X = X\}$$
$$\mathfrak{e} = \{X \in \mathfrak{g} : \Theta X = -X\}$$

for the corresponding eigenspaces.

The prominent example of effective orthogonal symmetric Lie algebra is the pair $(\mathfrak{g}, \Theta)$ coming from a globally Riemannian symmetric space (see Theorem II.11).

Lemma II.28

(1) *The decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is orthogonal with respect to the Killing form $B_{\mathfrak{g}}$.*

(2) *If $\mathfrak{g}$ is effective, $B_{\mathfrak{g}}\big|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite.*

Proof. (1) Let $X \in \mathfrak{u}$ and $Y \in \mathfrak{e}$ be arbitrary, so that, by definition, $\Theta X = X$ and $\Theta Y = -Y$. Moreover, since Θ is a Lie algebra automorphism,

$$B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(\Theta X, \Theta Y) = B_{\mathfrak{g}}(X, -Y)\,,$$

which implies that $B_{\mathfrak{g}}(X, Y) = 0$.

(2) Let $\langle , \rangle$ be an inner product on $\mathfrak{g}$ that is U-invariant. Therefore $U \subset O(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and $\mathrm{Lie}(U) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{o}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, that is, elements in $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ are skew-symmetric with regard to $\langle \cdot, \cdot \rangle$. Thus if $X \in \mathfrak{u}$ and $\{e_1, ..., e_n\}$ is a basis of $\mathfrak{g}$, then

$$\begin{aligned} B_{\mathfrak{g}}(X, X) &= \mathrm{Tr}\big(\mathrm{ad}_{\mathfrak{g}}(X)^2\big) \\ &= \sum_{j=0}^{n} \langle \mathrm{ad}_{\mathfrak{g}}(X)^2 e_j, e_j \rangle \\ &= -\sum_{j=1}^{n} \langle \mathrm{ad}_{\mathfrak{g}}(X) e_j, \mathrm{ad}_{\mathfrak{g}}(X) e_j \rangle \\ &= -\sum_{j=1}^{n} \|\mathrm{ad}_{\mathfrak{g}}(X) e_j\|^2 \\ &\leq 0 \end{aligned}$$

where we have equality if and only if $\mathrm{ad}_{\mathfrak{g}}(X) = 0$, that is $X \in \mathfrak{u} \cap \mathfrak{z}(\mathfrak{g}) = (0)$. ∎

Definition: Compact, Non-compact and Euclidean Type

Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra with Killing form $B_{\mathfrak{g}}$, and let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ be the decomposition of $\mathfrak{g}$ into the eigenspaces of Θ corresponding respectively to the $+1$ and the -1 eigenvalue.

(1) $(\mathfrak{g}, \Theta)$ is of **compact type** if $B_{\mathfrak{g}}$ is negative definite.

(2) $(\mathfrak{g}, \Theta)$ is of **non-compact type** if $B_{\mathfrak{g}}\big|_{\mathfrak{e}}$ is positive definite.

(3) $(\mathfrak{g}, \Theta)$ is of **Euclidean type** if $\mathfrak{e}$ is an Abelian ideal.

Recall. (1) The Killing form $B_{\mathfrak{g}}$ restricted to $\mathfrak{u}$ is negative definite, since $\mathfrak{u}$ is compactly embedded.

(2) A Lie algebra $\mathfrak{g}$ is **simple** if

- $\mathfrak{g}$ is not Abelian and
- $\mathfrak{g}$ contains no non-trivial ideals.

(3) A Lie algebra $\mathfrak{g}$ is **semisimple** if it is the direct sum of simple ideals:

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}_j.$$

Recall also that $\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is non-degenerate.

Remark. In cases (1) and (2) of the Definition, $\mathfrak{g}$ is semisimple.
Moreover, $(\mathfrak{g}, \Theta)$ is of Euclidean type if and only if $[\mathfrak{e}, \mathfrak{e}] = 0$.

The following will be needed for the proof of Theorem II.41. A proof is given in Marcs Notes Chapter IV, part 1.

Proposition II.29

If $\mathfrak{g}$ is semisimple, then

$$Der(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}).$$

In other words, every derivation of $\mathfrak{g}$ is inner.

We say that a pair (G, U) is **associated with an orthogonal symmetric Lie algebra** $(\mathfrak{g}, \Theta)$, if G is a connected Lie group with Lie algebra $\mathfrak{g}$, and U is a Lie subgroup of G with Lie algebra $\mathfrak{u}$. So one can define the type of a pair (G, U), according to the type of the effective orthogonal Lie algebra to which it is associated. Similarly, the type of a globally symmetric space M is defined as the type of an associated symmetric pair (G, K) naturally associated to an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ as above.

Notice that, even though every choice of a base point gives rise *a priori* to a different Riemannian symmetric pair, the types of such pairs are not changed: if instead of a base point $o \in M$ we take the base point $x = g \cdot o$, for $g \in G$, then the Lie algebra $\mathfrak{g}$ is the same and the involution Θ is replaced by $\mathrm{Ad}_{\mathrm{G}}(g)\Theta$.

Theorem II.30: Decomposition Theorem for OSLA

Let $(\mathfrak{g}, \Theta)$ *be an effective orthogonal symmetric Lie algebra. Then*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

is a decomposition into Θ*-invariant ideals such that*

(1) $\left(\mathfrak{g}_-, \Theta\big|_{\mathfrak{g}_-}\right)$ *is of non-compact type.*

(2) $\left(\mathfrak{g}_0, \Theta\big|_{\mathfrak{g}_0}\right)$ *is of Euclidean type.*

(3) $\left(\mathfrak{g}_+, \Theta\big|_{\mathfrak{g}_+}\right)$ *is of compact type.*

Moreover, the decomposition is orthogonal with regard to $B_{\mathfrak{g}}$*.*

How to construct $\mathfrak{g}_+, \mathfrak{g}_0, \mathfrak{g}_-$? Note that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is a U-invariant decomposition. Let then $\langle \cdot, \cdot \rangle$ be a U-invariant inner product on $\mathfrak{e}$. Since $B_{\mathfrak{g}}\big|_{\mathfrak{e}}$ is a symmetric bilinear form, there exists a unique $A \in \mathrm{End}(\mathfrak{e})$ symmetric such that

$$B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle \qquad \forall X, Y \in \mathfrak{e},$$

As $U \subset \mathrm{Aut}(\mathfrak{g})$ and $B_{\mathfrak{g}}$ is U-invariant, we note that if $X, Y \in \mathfrak{e}$ and $k \in U$ are arbitrary, then

$$B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(kX, kY) \iff \langle AX, Y \rangle = \langle AkX, kY \rangle = \langle k^{-1}AkX, Y \rangle,$$

hence $Ak = kA$ and therefore

$$A \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X) \circ A \qquad \forall X \in \mathfrak{u}.$$

As A is symmetric, there exists an orthonormal basis of $\mathfrak{e}$ which we write $\{f_1, ..., f_n\}$ consistingo f eigenvectors of A with eigenvalues $\{\beta_1, ..., \beta_n\}$. By the above property $Ak = kA$, they are also preserved by U and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$.

Let us define

$$\mathfrak{e}_- = \sum_{\beta_j < 0} \mathbb{R} f_j\,, \qquad \mathfrak{e}_0 = \sum_{\beta_j = 0} \mathbb{R} f_j\,, \qquad \mathfrak{e}_+ = \sum_{\beta_j > 0} \mathbb{R} f_j\,. \tag{II.17}$$

Lemma II.31

The subspaces $\mathfrak{e}_0, \mathfrak{e}_+$ and $\mathfrak{e}_-$ satisfy the following relations:

(1) $\mathfrak{e}_0 = \{X \in \mathfrak{g} : B_{\mathfrak{g}}(X,Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$.

(2) $[\mathfrak{e}_0, \mathfrak{e}] = \{0\}$ *and* $\mathfrak{e}_0$ *is an Abelian ideal in* $\mathfrak{g}$.

(3) $[\mathfrak{e}_-, \mathfrak{e}_+] = \{0\}$.

Proof. (1) Write

$$\mathfrak{g}^\perp = \{X \in \mathfrak{g} : B_{\mathfrak{g}}(X,Y) = 0 \ \forall Y \in \mathfrak{g}\}$$

and note that $\mathfrak{g}^\perp$ is Θ-invariant since $B_{\mathfrak{g}}$ is Θ-invariant. Thus we have a decomposition of $\mathfrak{g}^\perp$ induced by the one of $\mathfrak{g}$

$$\mathfrak{g}^\perp = (\mathfrak{g}^\perp \cap \mathfrak{u}) \oplus (\mathfrak{g}^\perp \cap \mathfrak{e})$$

As $(\mathfrak{g}, \Theta)$ is effective, $B_{\mathfrak{g}}\big|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite and therefore $\mathfrak{g}^\perp \cap \mathfrak{u} = (0)$ implying that $\mathfrak{g}^\perp \subset \mathfrak{e}$. Therefore

$$\begin{aligned}
\mathfrak{g}^\perp &= \{X \in \mathfrak{e} : B_{\mathfrak{g}}(X,Y) = 0 \ \forall Y \in \mathfrak{g}\} \\
&\overset{(II.28)}{=} \{X \in \mathfrak{e} : B_{\mathfrak{g}}(X,Y) = 0 \ \forall Y \in \mathfrak{e}\} \\
&= \{X \in \mathfrak{e} : \langle AX, Y\rangle = 0 \ \forall Y \in \mathfrak{e}\} \\
&= \ker(A) \\
&= \mathfrak{e}_0
\end{aligned}$$

by definition.

(2) Note first that $[\mathfrak{e}_0, \mathfrak{e}] \subset [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{u}$. Take then $X \in \mathfrak{e}_0, Y \in \mathfrak{e}, Z \in \mathfrak{u}$ and write

$$\begin{aligned}
B_{\mathfrak{g}}([X,Y],Z) &= -B_{\mathfrak{g}}([Y,X],Z) \\
&= -(-B_{\mathfrak{g}}(X,[Y,Z])) \\
&= \langle AX, [Y,Z]\rangle \\
&= 0
\end{aligned}$$

since $A \in \mathfrak{e}_0 = \ker(A)$. But $B_{\mathfrak{g}}\big|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite and as $Z \in \mathfrak{u}$ is arbitrary we must have $[X,Y] = 0$ for every $X \in \mathfrak{e}_0, Y \in \mathfrak{e}$. In particular thus $[\mathfrak{e}_0, \mathfrak{e}_0] = 0$ showing that $\mathfrak{e}_0$ is Abelian. Finally we note that,

$$[\mathfrak{e}_0, \mathfrak{g}] = [\mathfrak{e}_0, \mathfrak{u}] + [\mathfrak{e}_0, \mathfrak{e}] = [\mathfrak{e}_0, \mathfrak{u}] = \mathfrak{e}_0$$

because $\mathfrak{u}$ commutes with A and therefore preserves $\mathfrak{e}_-, \mathfrak{e}_0, \mathfrak{e}_+$.

(3) Take $X \in \mathfrak{e}_-$, $Y \in \mathfrak{e}_+$ and $Z \in \mathfrak{u}$. Then

$$\begin{aligned} B_{\mathfrak{g}}([X,Y],Z) &= -B_{\mathfrak{g}}(Y,[X,Z]) \\ &= -\langle \underbrace{AY}_{\in \mathfrak{e}_+}, \underbrace{[X,Z]}_{\in \mathfrak{e}_-} \rangle \\ &= 0 \end{aligned}$$

as $\mathfrak{e}_-$, $\mathfrak{e}_+$ are orthogonal with regard to $\langle \cdot, \cdot \rangle$ since they are defined using an orthonormal basis. As before, this implies

$$[X,Y] = 0 \qquad \forall X \in \mathfrak{e}_-,\ Y \in \mathfrak{e}_+ .$$ ■

We now define

$$\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+] \qquad \mathfrak{u}_- := [\mathfrak{e}_-, \mathfrak{e}_-] \qquad \text{and } \mathfrak{u}_0 := \mathfrak{u} \ominus_{B_{\mathfrak{g}}} (\mathfrak{u}_+ \oplus \mathfrak{u}_-)\,,$$

where the last equality denotes the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$ with respect to $B_{\mathfrak{g}}$.

Lemma II.32

The subspaces $\mathfrak{u}_0, \mathfrak{u}_+, \mathfrak{u}_-$ are orthogonal with respect to $B_{\mathfrak{g}}$ and their direct sum is $\mathfrak{u}$.

Proof. To see that $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are orthogonal with respect to $B_{\mathfrak{g}}$, let $X_\pm, Y_\pm \in \mathfrak{e}_\pm$. Then, by $\mathrm{ad}_{\mathfrak{g}}$-invariance of $B_{\mathfrak{g}}$, we have

$$B_{\mathfrak{g}}([X_+,Y_+],[X_-,Y_-]) = B_{\mathfrak{g}}(X_+,[Y_+,[X_-,Y_-]) = 0\,,$$

where the last equality follows from the Jacobi identity via

$$[Y_+,[X_-,Y_-] = -[X_-,[Y_-,Y_+]] - [Y_-,[Y_+,X_-]] = -[X_-,0] - [Y_-,0] = 0\,.$$ ■

Lemma II.33

We have:

(1) $\mathfrak{u}_\varepsilon$ *are ideals in* $\mathfrak{u}$ *that are pairwise orthogonal with regard to* $B_{\mathfrak{g}}$.

(2) $[\mathfrak{u}_0, \mathfrak{e}_-] = [\mathfrak{u}_0, \mathfrak{e}_+] = \{0\}$.

(3) $[\mathfrak{u}_-, \mathfrak{e}_0] = [\mathfrak{u}_-, \mathfrak{e}_+] = \{0\}$.

(4) $[\mathfrak{u}_+, \mathfrak{e}_0] = [\mathfrak{u}_+, \mathfrak{e}_-] = \{0\}$.

Proof. (1) We have already proved orthogonality so we are are just left to prove that they are ideals.

$$\begin{aligned}[\mathfrak{u}_\pm, \mathfrak{u}] &= [[\mathfrak{e}_\pm, \mathfrak{e}_\pm], \mathfrak{u}] \\ &\overset{\text{Jacobi}}{=} -[[\mathfrak{e}_\pm, \mathfrak{u}], \mathfrak{e}_\pm] - [[\mathfrak{u}, \mathfrak{e}_\pm], \mathfrak{e}_\pm] \\ &\overset{\mathfrak{e}_\pm\ \mathfrak{u}\text{-inv.}}{\subset} [\mathfrak{e}_\pm, \mathfrak{e}_\pm] \\ &= \mathfrak{u}_\pm\,.\end{aligned}$$

Let then $X \in \mathfrak{u}_0 \perp (\mathfrak{u}_+ \oplus \mathfrak{u}_-)$, $Z \in \mathfrak{u}$. We show

$$[Z, X] \perp (\mathfrak{u}_+ \oplus \mathfrak{u}_-)$$

that is, $\forall Y \in \mathfrak{u}_+ \oplus \mathfrak{u}_-$ we have

$$B_\mathfrak{g}([Z, X], Y) = -B_\mathfrak{g}(X, \underbrace{[Z, Y]}_{\in \mathfrak{u}_+ \oplus \mathfrak{u}_-}) = 0$$

(2) Let $Z \in \mathfrak{u}_0$, $X, Y \in \mathfrak{e}_\pm$. Then

$$B_\mathfrak{g}([Z, X], Y) = B_\mathfrak{g}(Z, [X, Y]) = 0\,,$$

since $[X, Y] \in \mathfrak{u}_\pm$ and $\mathfrak{u}_\pm$ is orthogonal to $\mathfrak{u}_0$. Since $[\mathfrak{u}_0, \mathfrak{e}_\pm] \subset \mathfrak{e}_\pm$ and $B_\mathfrak{g}$ restricted to $\mathfrak{e}_\pm$ is non-degenerate, then $[Z, X] = 0$, that is $[\mathfrak{u}_0, \mathfrak{e}_\pm] = \{0\}$.

(3) & (4) Using the definition of $\mathfrak{u}_\pm$ and the Jacobi identity, we have

$$\begin{aligned}[\mathfrak{u}_\pm, \mathfrak{e}_0] &= [[\mathfrak{e}_\pm, \mathfrak{e}_\pm], \mathfrak{e}_0] \\ &\overset{\text{Jacobi}}{=} [\mathfrak{e}_\pm, [\mathfrak{e}_\pm, \mathfrak{e}_0]] \\ &= \{0\},\end{aligned}$$

because of Lemma II.31 (2). Likewise,

$$\begin{aligned}[\mathfrak{u}_\pm, \mathfrak{e}_\mp] &= [[\mathfrak{e}_\pm, \mathfrak{e}_\pm], \mathfrak{e}_\mp] \\ &\overset{\text{Jacobi}}{=} [\mathfrak{e}_\pm, [\mathfrak{e}_\pm, \mathfrak{e}_\mp]] \\ &= \{0\},\end{aligned}$$

because of Lemma II.31 (3).

■

Now it is clear that since

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e} = (\mathfrak{u}_0 \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-) \oplus (\mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-)\,,$$

to find the $\mathfrak{g}_0, \mathfrak{g}_+$ and $\mathfrak{g}_-$ we have to rearrange the direct summands. It seems that setting

$$\mathfrak{g}_0 = \mathfrak{u}_0 \oplus \mathfrak{e}_0 \qquad \mathfrak{g}_+ = \mathfrak{u}_+ \oplus \mathfrak{e}_+ \qquad \mathfrak{g}_- = \mathfrak{u}_- \oplus \mathfrak{e}_-$$

might be a good idea.

Corollary II.34

$\mathfrak{u}_+ \oplus \mathfrak{e}_+$, $\mathfrak{u}_0 \oplus \mathfrak{e}_0$ and $\mathfrak{u}_- \oplus \mathfrak{e}_-$ are pairwise orthogonal ideals in $\mathfrak{g}$ with regard to $B_\mathfrak{g}$.

Proof. $\mathfrak{u} = \mathfrak{u}_- \oplus \mathfrak{u}_0 \oplus \mathfrak{u}_+$ and $\mathfrak{e} = \mathfrak{e}_- \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_+$ are both orthogonal with regard to $B_\mathfrak{g}$ so $\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon$ are pairwise orthogonal.

We show that they are ideals.

- To see that $\mathfrak{u}_0 \oplus \mathfrak{e}_0$ is an ideal, we only need to see what happens for $[\mathfrak{u}_0, \mathfrak{e}]$ as

  $$[\mathfrak{u}_0 \oplus \mathfrak{e}_0, \mathfrak{u} \oplus \mathfrak{e}] = \underbrace{[\mathfrak{u}_0, \mathfrak{u}]}_{\in \mathfrak{u}_0 \text{ by II.33}} + [\mathfrak{u}_0, \mathfrak{e}] + \underbrace{[\mathfrak{e}_0, \mathfrak{u}] + [\mathfrak{e}_0, \mathfrak{e}]}_{\in \mathfrak{e}_0 \text{ by II.31}}.$$

  But by Lemma II.33

  $$[\mathfrak{u}_0, \mathfrak{e}] = [\mathfrak{u}_0, \mathfrak{e}_0]$$

  and $\mathfrak{u}$ (thus in particular $\mathfrak{u}_0$) preserves the decomposition of $\mathfrak{e}$. Therefore

  $$[\mathfrak{u}_0, \mathfrak{e}_0] \subset \mathfrak{e}_0$$

  and hence $\mathfrak{u}_0 \oplus \mathfrak{e}_0$ is an ideal.

- To see that $\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon$ is an ideal, note that

  $$[\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon, \mathfrak{u} \oplus \mathfrak{e}] = \underbrace{[\mathfrak{u}_\varepsilon, \mathfrak{u}]}_{\in \mathfrak{e}_\varepsilon} + \underbrace{[\mathfrak{u}_\varepsilon, \mathfrak{e}]}_{= [\mathfrak{u}_\varepsilon, \mathfrak{e}_\varepsilon]} + \underbrace{[\mathfrak{e}_\varepsilon, \mathfrak{u}]}_{\in \mathfrak{e}_\varepsilon} + \underbrace{[\mathfrak{e}_\varepsilon, \mathfrak{e}]}_{\subset [\mathfrak{e}_\varepsilon, \mathfrak{e}_\varepsilon] \subset \mathfrak{u}_\varepsilon}.$$ ■

Summary We have for $(\mathfrak{g}, \Theta)$ an (effective) orthogonal symmetric Lie algebra the decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{u} \oplus \mathfrak{e} \\ &= (\mathfrak{u}_- \oplus \mathfrak{u}_0 \oplus \mathfrak{u}_+) \oplus (\mathfrak{e}_- \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_+) \\ &= \underbrace{(\mathfrak{u}_- \oplus \mathfrak{e}_-)}_{= \mathfrak{g}_-} \oplus \underbrace{(\mathfrak{u}_0 \oplus \mathfrak{e}_0)}_{= \mathfrak{g}_0} \oplus \underbrace{(\mathfrak{u}_+ \oplus \mathfrak{e}_+)}_{= \mathfrak{g}_+} \end{aligned}$$

such that

(1) $\mathfrak{g}_0, \mathfrak{g}_+$ and $\mathfrak{g}_-$ are pairwise orthogonal ideals in $\mathfrak{g}$, so that in particular their Killing form is the restriction of the Killing form of $\mathfrak{g}$ i.e. $B_{\mathfrak{g}_\varepsilon} = B_{\mathfrak{g}}\big|_{\mathfrak{g}_\varepsilon \times \mathfrak{g}_\varepsilon}$.

(2) $(\mathfrak{g}, \Theta)$ is effective and therefore $B_{\mathfrak{g}}\big|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite. Thus

- As $B_{\mathfrak{g}}\big|_{\mathfrak{e}_-\times\mathfrak{e}_-}$ is negative definite, $B_{\mathfrak{g}_-}$ is negative definite and $\mathfrak{g}_-$ is therefore of compact type.
- As $B_{\mathfrak{g}}\big|_{\mathfrak{e}_+\times\mathfrak{e}_+}$ is positive definite, $B_{\mathfrak{g}_+}$ is non-degenerate and $\mathfrak{g}_+$ is therefore of non-compact type.

In both cases, $\mathfrak{g}_\pm$ is semisimple.

(3) We showed in Lemma II.31 (2) that $\mathfrak{e}_0$ is an Abelian ideal. Moreover, since $\mathfrak{g}_\pm$ are semisimple, the center $\mathfrak{z}$ of $\mathfrak{g}$ must be all contained in $\mathfrak{g}_0$ and hence

$$\mathfrak{z}(\mathfrak{g}_0) = \mathfrak{z}(\mathfrak{g}).$$

Thus

$$\mathfrak{z}(\mathfrak{g}_0) \cap \mathfrak{u}_0 \subset \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{u} = 0$$

and hence we are left to observe that $\mathfrak{u}_0$ is compactly embedded. But this is true since $\mathfrak{u} \subset \mathfrak{g}$, $\mathfrak{u}_\pm \subset \mathfrak{g}_\pm$ are all compactly embedded and $\mathfrak{g}$ is the direct sum of the ideals $\mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$, (see [Hel01, Lemma V.1.6]). Hence $(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0})$ is an effective orthogonal symmetric Lie algebra.

Remark. We were a bit sloppy in the last part of the proof, in that the decomposition we proposed is valid only if $\mathfrak{e}_0 \neq \{0\}$. In fact, if $\mathfrak{e}_0 = \{0\}$, then our proposed $\mathfrak{g}_0$ would be equal to $\mathfrak{u}_0$. As a consequence, we would have that $\Theta = Id$, which was not allowed. We hence set if $\mathfrak{e}_0 = \{0\}$:

$$\begin{array}{llll} \mathfrak{g}_0 \coloneqq \{0\} & \mathfrak{g}_- \coloneqq \mathfrak{u}_0 \oplus \mathfrak{u}_- \oplus \mathfrak{e}_- & \mathfrak{g}_+ \coloneqq \mathfrak{u}_+ \oplus \mathfrak{e}_+ & \text{if } \mathfrak{e}_- \neq \{0\}\,; \\ \mathfrak{g}_0 \coloneqq \{0\} & \mathfrak{g}_- \coloneqq \{0\} & \mathfrak{g}_+ \coloneqq \mathfrak{u}_- \oplus \mathfrak{u}_+ \oplus \mathfrak{e}_+ & \text{if } \mathfrak{e}_- = \{0\}\,. \end{array}$$

Remark. If (G, K) is a Riemannian symmetric pair, then we have an associated orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ with $\mathfrak{k}$ compactly embedded. Since

$$\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = \mathrm{Lie}(Z(G) \cap K)$$

we note that $(\mathfrak{g}, \Theta)$ is effective if and only if $Z(G) \cap K$ is discrete.

Definition: Effective Riemannian Symmetric Pairs

A Riemannian Symmetric Pair (G, K) is **effective** if $Z(G) \cap K$ is discrete.

Lemma II.35

Let M be a Riemannian symmetric space, $G = \mathrm{Iso}(M)^\circ$, $o \in M$ and $K = \mathrm{Stab}_G(o)$. If $N \triangleleft G$ is contained in K, $N = \{e\}$ and in particular the Riemannian symmetric pair (G, K) is effective.

Proof. If $g_* o \in M$, then $\mathrm{Stab}_G(g_* o) = gKg^{-1}$. Since $N \triangleleft G$ and $N < K$ we have

$$N \subset \bigcap_{g \in G} gKg^{-1} = \bigcap_{g \in G} \mathrm{Stab}_G(g_* o) = \{e\}.$$

Since every subgroup of $Z(G)$ is normal, (G, K) is effective. ∎

Definition: Riemannian Symmetric Spaces of Compact, Non-compact and Euclidean Type

- An effective Riemannian symmetric pair (G, K) is of **compact, non-compact or euclidean type** if the corresponding orthogonal symmetric Lie algebra is.
- A Riemannian symmetric space M is of **compact, non-compact or euclidean type** if the corresponding Riemannian symmetric pair $(\mathrm{Iso}(M)^\circ, \mathrm{Stab}_{\mathrm{Iso}(M)^\circ}(o))$ is.

Theorem II.36

If M is a simply connected Riemannian symmetric space, then M is a Riemannian product

$$M = M_- \times M_0 \times M_+$$

where

- *M_- is of compact type,*
- *M_0 is of euclidean type and*
- *M_+ is of non-compact type.*

Proof. CHECK!! Write $G = \mathrm{Iso}(M)^\circ$, $o \in M$, $\sigma = s_o g s_o$ and $\Theta = \mathrm{d}_e\sigma$. Then $(\mathfrak{g}, \Theta)$ is an orthogonal symmetric Lie algebra which can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

Let then G_ε be the Lie subgroups of G corresponding to $\mathfrak{g}_\varepsilon$. Since the $\mathfrak{g}_\varepsilon$ are ideals, the G_ε are normal subgroups and $G_\varepsilon \cap G_\eta$ is discrete for $\varepsilon \neq \eta$. We claim that

$[G_\varepsilon, G_\eta] = 0$ and that

$$\varphi\colon G_- \times G_0 \times G_+ \to G$$
$$(x, y, z) \mapsto xyz$$

is a homomorphism. In fact, $[G_\varepsilon, G_\eta] \lhd G_\varepsilon \cap G_\eta$ but $[G_\varepsilon, G_\eta]$ is connected and hence trivial. Now we observe that if

$$\mathrm{d}_e\varphi\colon \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \to \mathfrak{g}$$

is an isomorphism, then

$$\tilde{\varphi}\colon \tilde{G}_- \times \tilde{G}_0 \times \tilde{G}_+ \to \tilde{G}$$

is an isomorphism as well. Let $p\colon \tilde{G} \to G$ be the projection. Then

$$\tilde{G}/(p^{-1}(K))^\circ \to \tilde{G}/p^{-1}(K) = G/K = M$$

as M is simply connected and thus $p^{-1}(K) < \tilde{G}$ is connected.

$$\mathfrak{k} = \mathrm{Lie}(p^{-1}(K)) \subset \mathfrak{g}$$

and let $\mathfrak{k}_\varepsilon$ be the subalgebras $\mathfrak{k} = \mathfrak{k}_- \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_+$ and K_ε the corresponding subgroups of $\tilde{G}$.

$$\tilde{\varphi}(K_- \times K_0 \times K_+) = p^{-1}(K)$$

and the K_ε are closed, hence $(\tilde{G}_\varepsilon, K_\varepsilon)$ are a Riemannian symmetric pair with regard to the lift $\tilde{\sigma}$ of σ. Thus

$$\tilde{\varphi}\colon \tilde{G}_-/K_- \times \tilde{G}_0/K_0 \times \tilde{G}_+/K_+ \to M$$

is a diffeomorphism. ■

II.9.2 Irreducible orthogonal Symmetric Lie Algebras

Definition: Reduced orthogonal symmetric Lie algebras

An orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ is **reduced** if $\mathfrak{u}$ does not contain any non-zero ideals.

Remark. If $\mathfrak{n} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{n} \subset \mathfrak{u}$ is trivial if and only if any connected normal subgroup $N \lhd G$ contained in K, $N < K$, is trivial.

In particular, reduced therefore implies that

$$\bigcap_{g \in G} gKg^{-1} = \bigcap_{g \in G} \mathrm{Stab}_G(g_* o)$$

can not be connected and is thus discrete. Also the action of G on G/K has discrete kernel.

Hence, if $(\mathfrak{g}, \Theta)$ is reduced, it is also effective. In fact, $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{u}$ is a subalgebra of $\mathfrak{z}(\mathfrak{g})$ and since the latter is Abelian it is actually an ideal in $\mathfrak{g}$ contained in $\mathfrak{u}$. As $(\mathfrak{g}, \Theta)$ is reduced, it must thus be trivial.

Definition: Irreducible orthogonal Symmetric Lie Algebra

Let $(\mathfrak{g}, \Theta)$ be an orthogonal symmetric Lie algebra (with decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$). We say that $(\mathfrak{g}, \Theta)$ is **irreducible** if

(1) $\mathfrak{g}$ is semisimple and $(\mathfrak{g}, \Theta)$ is reduced, and

(2) $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ acts irreducibly on $\mathfrak{e}$.

Theorem II.37

A reduced orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ is the direct sum of irreducible orthogonal symmetric Lie algebras and the decomposition is unique.

Theorem II.38

Let V be a real vector space and $K < \mathrm{GL}(V)$ compact. Then there exists a decomposition $V = \bigoplus_i V_i$ into K-invariant irreducible subspaces.

Proof of Theorem II.38. Let $\langle \cdot, \cdot \rangle$ be a K-invariant inner product on V. If V is irreducible, we are done. If not, take a K-invariant subspace and note that $W^\perp$ is also invariant. ■

Sketch of Proof of Theorem II.37. Let $\mathfrak{u}, \mathfrak{e}$ be as before and $\langle \cdot, \cdot \rangle$ an inner product on $\mathfrak{e}$ that is $\mathfrak{u}$-invariant. Let $A \in \mathrm{End}(\mathfrak{e})$ be symmetric such that

$$B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle \qquad \forall X, Y \in \mathfrak{e}.$$

Let then $\mathfrak{e} = \bigoplus_{i=1}^r \mathfrak{q}_i$ be the decomposition corresponding to the distinct eigenvalues $c_0 = 0, c_1, ..., c_r \neq 0$ with $c_i \neq c_j$ for $1 \leq i \neq j \leq r$. This decomposition is also $\mathfrak{u}$-invariant. By the previous theorem, we can decompose the $\mathfrak{q}_i$ into $\mathfrak{u}$-invariant irreducible subspaces

$$\mathfrak{q}_i = \bigoplus_{j=1}^{r_i} \mathfrak{p}_{ij}$$

The $\mathfrak{p}_{ij}$ play the role of $\mathfrak{p}$ in the Cartan decomposition.

Define

$$\mathfrak{g}_{ij} \coloneqq [\mathfrak{p}_{ij}, \mathfrak{p}_{ij}] + \mathfrak{p}_{ij}$$

and show that

(1) The $\mathfrak{g}_{ij}$ are Θ-invariant ideals in $\mathfrak{g}$,

(2) $B_{\mathfrak{g}_{ij}} = B_{\mathfrak{g}}\big|_{\mathfrak{g}_{ij}\times\mathfrak{g}_{ij}}$ is non-degenerate and

(3) $\mathfrak{m} := \bigoplus \mathfrak{g}_{ij}$ is a semisimple Θ-invariant ideal and $\mathfrak{g}_0 = \mathrm{Centr}_{\mathfrak{g}}(\mathfrak{m})$. This is also Θ-invariant and $(\mathfrak{g}_0, \Theta\big|_{\mathfrak{g}_0})$ is a euclidean orthogonal symmetric Lie algebra. ■

Definition: Reduced/Irreducible Riemannian Symmetric Spaces

- A Riemannian symmetric pair (G, K) is **reduced or irreducible** if the corresponding orthogonal symmetric Lie algebra is.
- A Riemannian symmetric space M is **reduced or irreducible** if the corresponding Riemannian symmetric pair $(\mathrm{Iso}(M)^\circ, \mathrm{Stab}_{\mathrm{Iso}(M)^\circ}(o))$ is.

Remark. A Riemannian symmetric space is irreducible if $\mathrm{Lie}(\mathrm{Iso}(M)^\circ)$ is semisimple, K acts irreducibly via Ad_G on $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition.

Corollary II.39

A Riemannian symmetric space M is isometric to the Riemannian product $M = M_0 \times \ldots \times M_n$ where

- $M_o = \mathbb{E}^k$ *and*
- $M_i, 1 \leq i \leq n$ *are irreducible symmetric spaces of compact or non-compact type.*

Remark. M being irreducible does not imply that $\mathrm{Iso}(M)^\circ$ is simple.

For example, let U be a compact Lie group and consider the Riemannian symmetric pair $(U \times U, \Delta U)$ with $\Delta(U) = \{(g, g) \in U \times U\}$. Define then

$$\Theta(X, Y) = (Y, X)$$

and note that this implies

$$\mathfrak{k} = \{(X, X) : X \in \mathfrak{u}\}$$
$$\mathfrak{p} = \{(Y, -Y) : Y \in \mathfrak{u}\}$$

with the $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$-action

$$(\mathrm{ad}_{\mathfrak{g}}(X, X))(Y, -Y) = [(X, X), (Y, -Y)] = ([X, Y], -[X, Y]).$$

If U is simple, then $U \times U/\Delta(U)$ is an irreducible symmetric space.

Proposition II.40

Let (G,K) be an irreducible Riemannian symmetric pair with Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ and let $B_{\mathfrak{g}}$ be the Killing form. Then there exist a (up to scalars) unique G-invariant Riemannian metric on G/K and on of the following holds:

(1) $B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}} \gg 0$ is positive definite, G/K is of non-compact type and the Riemannian metric is $B_{\mathfrak{g}}$.

(2) $B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}} \ll 0$ is negative definite, G/K is of compact type and the Riemannian metric is $-B_{\mathfrak{g}}$.

Proof. Take an $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ and as previously

$$B_{\mathfrak{g}}(X,Y)=\langle AX,Y\rangle \qquad \forall X,Y\in\mathfrak{p}.$$

Since $\mathrm{Ad}_{\mathrm{G}}(K)$ acts irreducibly on $\mathfrak{p}$ we must have $A=\lambda Id_{\mathfrak{p}}$ for some $0\neq\lambda\in\mathbb{R}$. Whether $B_{\mathfrak{g}}$ is positive or negative definite then depends on the sign of λ. ■

II.10 From orthogonal Symmetric Lie Algebras to Riemannian Symmetric Spaces

In this subsection we want to apply the techniques developped so far to see how one can get from an orthogonal symmetric Lie algebra to a Riemannian symmetric space. The first step is to g(o) from a reduced semisimple orthogonal symmetric Lie algebra to a reduced semisimple Riemannian symmetric pair.

Theorem II.41

Let $(\mathfrak{g},\Theta)$ be a reduced semisimple orthogonal symmetric Lie algebra. Set $G:=\mathrm{Aut}(\mathfrak{g})^\circ$ and define the involution via

$$\begin{aligned}\sigma\colon G&\to G\\ \alpha&\mapsto\Theta\alpha\Theta^{-1}.\end{aligned}$$

Let $K<G$ such that $\left(G^\sigma\right)^\circ<K<G^\sigma$. Then (G,K) is a Riemannian symmetric pair whose associated orthogonal symmetric Lie algebra is isomorphic to $(\mathfrak{g},\Theta)$. Moreover, G^σ is compact, $Z(G)=\{e\}$ and G acts faithfully on G/K.

Proof. Using Theorem II.37 we can write

$$(\mathfrak{g}, \Theta) \cong \prod_{i=1}^{k} (\mathfrak{g}_i, \Theta_i)$$

as a product of irreducible components. It is left as an exercise to show that under this isomorphism

$$\operatorname{Aut}(\mathfrak{g})^\circ \xrightarrow{\sim} \prod_{i=1}^{k} \big(\operatorname{Aut}(\mathfrak{g}_i)\big)^\circ$$
$$G^\sigma \xrightarrow{\sim} \prod_{i=1}^{k} G^{\sigma_i}$$

and hence we may assume that $(\mathfrak{g}, \Theta)$ is irreducible.

Recall that $\Theta \in \operatorname{Aut}(\mathfrak{g})$ is an involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. We define a scalar product

$$\langle \cdot, \cdot \rangle$$

on $\mathfrak{g}$ as follows:

(1) If $\mathfrak{g}$ is of compact type we set $\langle X, Y \rangle = -B_{\mathfrak{g}}(X, Y)$ for $X, Y \in \mathfrak{g}$.

(2) If $\mathfrak{g}$ is of non-compact type we set

$$\langle X, Y \rangle = -B_{\mathfrak{g}}(X, \Theta(Y)) \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

As $\mathfrak{g}$ is semisimple it follows from Proposition II.29 that

$$\operatorname{ad}_{\mathfrak{g}} \colon \mathfrak{g} \to \operatorname{Der}\mathfrak{g} \subset \operatorname{Lie}(G)$$

is an isomorphism. Moreover, $G^\sigma = \{\alpha \in G : \alpha\Theta = \Theta\alpha\}$ is obviously a closed subgroup of G.

We claim that G^σ is compact. Indeed, G^σ preserves the scalar product $\langle \cdot, \cdot \rangle$. This is clear in the case when $\mathfrak{g}$ is of compact type, because $\operatorname{Aut}(\mathfrak{g})$ preserves the Killing form $B_{\mathfrak{g}}$ according to the reminder (1) at the beginning of section II.9.1.

If $\mathfrak{g}$ is of non-compact type, then for all $\alpha \in G^\sigma$ and all $X, Y \in \mathfrak{g}$

$$\langle \alpha X, \alpha Y \rangle = -B_{\mathfrak{g}}(\alpha X, \Theta\alpha Y) = -B_{\mathfrak{g}}(\alpha X, \alpha\Theta Y) = -B_{\mathfrak{g}}(X, \Theta Y) = \langle X, Y \rangle.$$

We next compute the Lie algebra of G^σ using again Proposition II.29. We have

$$\begin{aligned} \operatorname{Lie}(G^\sigma) &= \{D \in \operatorname{Der}(\mathfrak{g}) : D\Theta = \Theta D\} \\ &= \{\operatorname{ad}_{\mathfrak{g}}(X) : \Theta \operatorname{ad}_{\mathfrak{g}}(X) = \operatorname{ad}_{\mathfrak{g}}(X)\Theta\} \\ &= \{\operatorname{ad}_{\mathfrak{g}}(X) : \operatorname{ad}_{\mathfrak{g}}(\Theta X) = \operatorname{ad}_{\mathfrak{g}}(X)\} = \operatorname{ad}_{\mathfrak{g}} \mathfrak{k}, \end{aligned}$$

hence $\mathrm{Lie}(K) = \mathrm{ad}_{\mathfrak{g}}\,\mathfrak{k}$. So $\mathrm{ad}_{\mathfrak{g}}$ establishes an isomorphism between $(\mathfrak{g}, \Theta)$ and the orthogonal symmetric Lie algebra associated to the Riemannian symmetric pair (G, K).

In order to prove the last assertion we notice that (G, K) is reduced, so the kernel N of the G-action on G/K is discrete. Since G is connected and N is a discrete normal subgroup we obtain $N < Z(G)$.

We finally take $\alpha \in Z(G)$ arbitrary. Then for all $\beta \in \mathrm{Aut}(g)^\circ$ we have $\alpha\beta = \beta\alpha$. Passing to the Lie algebra this implies

$$\alpha\,\mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X)\alpha \quad \text{for all} \quad X \in \mathfrak{g}$$

or equivalently

$$\mathrm{ad}_{\mathfrak{g}}(\alpha X) = \mathrm{ad}_{\mathfrak{g}}(X) \quad \text{for all} \quad X \in \mathfrak{g},$$

which shows that $\alpha = Id_{\mathfrak{g}}$, hence $N = Z(G) = \{e\}$. ■

Given a Riemannian symmetric space $M = G/K$ with an effective G-action we obviously have $G < \mathrm{Iso}(M)^\circ$. We now address the question when we have equality. Clearly this is not always the case as the example of $G = \mathbb{R}^n$, $\sigma(v) = -v$ for $v \in \mathbb{R}^n$ shows: $M = \mathbb{E}^n$, but $\mathrm{Iso}(M)^\circ = \mathbb{R}^n \ltimes \mathrm{SO}(n)$. But these are essentially the only examples.

Theorem II.42

Let (G, K) be a Riemannian symmetric pair and assume that G is semisimple and acts faithfully on $M = G/K$. Then $G = \mathrm{Iso}(M)^\circ$ and $K = \mathrm{Stab}_G(o)$, where $o = eK$.

Proof. Let $G_0 = \mathrm{Iso}(M)^\circ$ and $\tau\colon G \to G_0$ given by

$$\tau(g)hK := ghK.$$

Then τ is a smooth injective homomorphism. Let $\sigma\colon G \to G$ be the involution such that $(G^\sigma)^\circ < K < G^\sigma$ and denote $\pi\colon G \twoheadrightarrow M = G/K$ the natural projection. If $s_o\colon M \to M$ denotes the geodesic symmetry at $o = eK$, then

$$s_o\pi = \pi s_o.$$

Define

$$\begin{aligned}\sigma_0\colon G_0 &\to G_0\\ \alpha &\mapsto s_o\alpha s_o^{-1}\end{aligned}$$

and $K_0 = \mathrm{Stab}_{G_0}(o)$. Then (G_0, K_0) is a Riemannian symmetric pair and it follows from $s_o\pi = \pi s_o$ that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ G_0 & \xrightarrow{\sigma_0} & G_0 \end{array}$$

commutes. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the orthogonal symmetric Lie algebra $(\mathfrak{g}, D_e\sigma)$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the one of the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, D_e\sigma_0)$. Then it follows from the commuting diagram above that

$$D_e\tau(\mathfrak{p}) \subset \mathfrak{p}_0, \quad D_e\tau(\mathfrak{k}) \subset \mathfrak{k}_0.$$

Since $M = G/K = G_0/K_0$ we have $\dim \mathfrak{p} = \dim \mathfrak{p}_0$ and hence $D_e\tau(\mathfrak{p}) = \mathfrak{p}_0$ by injectivity of $D_e\tau$.

Next we notice that the inclusion

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

from Proposition II.21 is true for any reduced semisimple orthogonal symmetric Lie algebra. Indeed, one can easily verify that $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is an ideal in $\mathfrak{g}$ and that its orthogonal complement with respect to $B_\mathfrak{g}$ is contained in the orthogonal complement of $\mathfrak{p}$ which is equal to $\mathfrak{g}$. Hence this orthogonal complement vanishes and we get $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$. From $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ we therefore get $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Combining this with $D_e\tau(\mathfrak{p}) = \mathfrak{p}_0$ we conclude

$$D_e\tau(\mathfrak{k}) = [\mathfrak{p}_0, \mathfrak{p}_0].$$

Next we observe that $(\mathfrak{g}_0, D_e\sigma_0)$ is a reduced and hence effective orthogonal symmetric Lie algebra and that the null space $\mathfrak{e}_0$ of the Killing form of $\mathfrak{g}_0$ is contained in $\mathfrak{p}_0$ and an Abelian ideal in $\mathfrak{g}_0$. This gives the inclusion

$$\mathfrak{e}_0 \subset \mathfrak{p}_0 = D_e\tau(\mathfrak{p}) \subset D_e\tau(\mathfrak{g}) = \mathfrak{g}',$$

so $\mathfrak{e}_0$ is an Abelian ideal in $D_e\tau(\mathfrak{g})$ as well. But as $\mathfrak{g} \cong D_e\tau(\mathfrak{g})$ is semisimple this implies $\mathfrak{e}_0 = 0$. So $(\mathfrak{g}_0, D_e\sigma_0)$ is a reduced semisimple orthogonal symmetric Lie algebra, hence in particular $[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0$ which implies

$$D_e\tau(\mathfrak{k}) = \mathfrak{k}_0.$$

So τ induces an isomorphism between the two orthogonal symmetric Lie algebras and is therefore a Lie group isomorphism. ■

II.11 Curvature

Definition: (Sectional) Curvature

Let M be a Riemannian manifold with Levi-Civita connection ∇. The **curvature** of M is a multilinear map

$$R\colon \operatorname{Vect}(M) \times \operatorname{Vect}(M) \times \operatorname{Vect}(M) \to \operatorname{Vect}(M)$$

where $\operatorname{Vect}(M)$ is a C^∞-module defined as

$$R(X,Y)Z \coloneqq \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

Remark. For $p \in M$ $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p.

From R one can define the **sectional curvature**: Let $p \in M$ and $\operatorname{Gr}_2(T_pM)$ be the Grassmannian of 2-planes in T_pM. Define then

$$\begin{aligned} \kappa_p\colon \operatorname{Gr}_2(T_pM) &\to \mathbb{R} \\ P &\mapsto \langle R(u,v)u, v\rangle \end{aligned}$$

where $\{u, v\}$ is an orthonormal basis of P.

Theorem II.43

Let (G,K) be a Riemannian symmetric pair with associated Riemannian symmetric space M and a corresponding G-invariant Riemannian metric.

(1) If (G,K) is of compact type, then $\kappa_p \geq 0$ for all $p \in M$.

(2) If (G,K) is of non-compact type, then $\kappa_p \leq 0$ for all $p \in M$.

(3) If (G,K) is of euclidean type, then $\kappa_p = 0$ for all $p \in M$.

The proof of this relies on the following result:

Theorem II.44

Let (G,K) be a symmetric pair a and let R be the curvature tensor. Then at the point $o \in G/K$

$$R_o(\bar{X}_1, \bar{X}_2)\bar{X}_3 = -[[\bar{X}_1, \bar{X}_2], \bar{X}_3]$$

where $\bar{X}_i = \mathrm{d}_e\pi X_i$, for $X_i \in \mathfrak{p}$, $i = 1,2,3$.

Proof of Theorem II.43. Take $X_1, X_2 \in \mathfrak{p}$. Then

$$B_{\mathfrak{g}}(-[[X_1, X_2], X_1], X_2) = B_{\mathfrak{g}}([X_1, X_2], [X_1, X_2])$$

We restrict to the first case and note that if (G, K) is of compact type, then we take $-B_{\mathfrak{g}}$ as the Riemannian metric at o after $\mathfrak{p} \cong T_oM$.

Let $X_1, X_2 \in \mathfrak{p}$ such that $\bar{X}_1, \bar{X}_2 \in T_oM$ are orthonormal. Then

$$\begin{aligned}
\kappa_o(\mathrm{Span}(\bar{X}_1, \bar{X}_2)) &= -\langle R(\bar{X}_1, \bar{X}_2)\bar{X}_1, \bar{X}_2\rangle && \text{(II.18)}\\
&\overset{II.44}{=} -\langle \overline{[[X_1, X_2], X_1]}, \bar{X}_2\rangle && \text{(II.19)}\\
&= -B_{\mathfrak{g}}([[X_1, X_2], X_1], X_2) && \text{(II.20)}\\
&= B_{\mathfrak{g}}([X_1, X_2], [X_1, X_2]) && \text{(II.21)}\\
&= \langle [X_1, X_2], [X_1, X_2]\rangle = \|[X_1, X_2]\|^2 \geq 0. \quad \blacksquare && \text{(II.22)}
\end{aligned}$$

II.12 Duality

There is a remarkable and important duality between compact and non-compact orthogonal symmetric Lie algebras which is a special case of a general construction we will outline now. First we need some preliminaries on complexifications of real vector spaces and real Lie algebras.

Definition: Complex structure

Let V be a real vector space and $\mathfrak{v}$ a real Lie algebra. A **complex structure on** V is given by

$$J \in \mathrm{End}(V) \qquad \text{such that} \qquad J^2 = -Id.$$

A **complex structure on** $\mathfrak{v}$ is a complex structure $J \in \mathrm{End}(\mathfrak{v})$ of $\mathfrak{v}$ as a vector space which in addition satisies

$$[X, JY] = J[X, Y].$$

Any real vector space V with a complex structure J can be turned into a complex vector space, denoted $\tilde{V}$, by setting

$$(\alpha + i\beta)v := \alpha v + \beta J(v)$$

Conversely, any complex vector space W can be considered as a real vector space, denoted $W^{\mathbb{R}}$, with a complex structure $J \in \mathrm{End}(W^{\mathbb{R}})$ given by $J(w) = i \cdot w$. Then obviously $\widetilde{W^{\mathbb{R}}} = W$.

If $J \in \operatorname{End}(\mathfrak{v})$ is a complex structure on $\mathfrak{v}$, then it follows that $[\cdot,\cdot]\colon \tilde{\mathfrak{v}} \times \tilde{\mathfrak{v}} \to \tilde{\mathfrak{v}}$ is $\mathbb{C}$-bilinear and $\tilde{\mathfrak{v}}$ is a $\mathbb{C}$-Lie algebra.

In general, real vector spaces do not always admit a complex structure. However, for any real vector space V one can define an endomorphism $J \in \operatorname{End}(V \times V)$ with $J^2 = -Id$ by

$$\begin{aligned} J\colon V \times V &\longrightarrow V \times V \\ (v,w) &\mapsto (-w,v). \end{aligned}$$

Definition: Complexification

The **complexification of** V is $V^{\mathbb{C}} := \widetilde{V \times V}$. The **complex conjugation** on $V^{\mathbb{C}}$ is the $\mathbb{R}$-linear automorphism $\tau \in \operatorname{End}(V \times V)$ defined by

$$\tau(v,w) = (v,-w).$$

V embeds into $V^{\mathbb{C}}$ as a real vector space by

$$\begin{aligned} V &\hookrightarrow V^{\mathbb{C}} \\ v &\mapsto (v,0) \end{aligned}$$

Notice that the map

$$\begin{aligned} V^{\mathbb{C}} &\xrightarrow{\cong} V + iV \\ (v,w) &\mapsto v + iw \end{aligned}$$

is $\mathbb{C}$-linear and bijective, hence $V^{\mathbb{C}}$ can be identified with $V + iV$. Then complex conjugation is defined as usual, namely by

$$\tau(v+iw) = v - iw.$$

Remark. If $\mathfrak{v}$ is a real Lie algebra, the Lie bracket on $\mathfrak{v}$ extends uniquely to a $\mathbb{C}$-linear Lie bracket on $\mathfrak{v}^{\mathbb{C}}$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. Then $\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$. In fact,

$$\begin{aligned} A \in \mathfrak{sl}(n,\mathbb{C}) &\iff \operatorname{tr}(A) = 0 \\ &\iff \operatorname{Re}\operatorname{tr}(A) = \operatorname{Im}\operatorname{tr}(A) = 0 \\ &\iff \operatorname{tr}\operatorname{Re}(A) = \operatorname{tr}\operatorname{Im}(A) = 0 \\ &\iff A = A_1 + iA_2,\ A_i \in \mathfrak{sl}(n,\mathbb{R}) \end{aligned}$$

and thus

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{R}) + i\mathfrak{sl}(n,\mathbb{R}).$$

Example. Let $\mathfrak{g} = \mathfrak{su}(n,\mathbb{C}) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : X^* + X = 0, \text{ where } X^* = \overline{X}^t\}$. Observe that $\mathfrak{su}(n,\mathbb{C})$ is a real Lie algebra. We claim that $\mathfrak{su}(n,\mathbb{C})^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$. In fact,

$$\begin{aligned} i\mathfrak{su}(n,\mathbb{C}) &= \{iX \in \mathfrak{sl}(n,\mathbb{C}) : X^* + X = 0\} \\ &= \{X \in \mathfrak{sl}(n,\mathbb{C}) : X^* = X\}. \end{aligned}$$

But for any $A \in \mathfrak{sl}(n,\mathbb{C})$ we can write

$$A = \underbrace{\frac{A - A^*}{2}}_{\in \mathfrak{su}(n,\mathbb{C})} + \underbrace{\frac{A + A^*}{2}}_{\in i\mathfrak{su}(n,\mathbb{C})},$$

so $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{su}(n,\mathbb{C}) \oplus i\mathfrak{su}(n,\mathbb{C})$ and a count of dimensions gives equality.

Example. Let $\mathfrak{g} = \mathfrak{o}(p,q)$. Since any two non-degenerate quadratic forms over $\mathbb{C}$ are equivalent, we have $\mathfrak{o}(p,q)^{\mathbb{C}} = \mathfrak{o}(p+q,\mathbb{C})$. In particular $\mathfrak{o}(n,\mathbb{R})^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C})$ and $\mathfrak{o}(1,n-1)^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C})$.

Definition: Complexificaton of endomorphisms

If V is a real vector space and $T \in \operatorname{End}(V)$, the map

$$\begin{aligned} V^{\mathbb{C}} = V + iV &\longrightarrow V + iV \\ v + iw &\mapsto Tv + iTw \end{aligned}$$

is an endomorphism of the $\mathbb{C}$-vector space $V^{\mathbb{C}}$.

Notice that if $T_1, T_2, T \in \operatorname{End}(V)$, then

$$(T_1 \circ T_2)^{\mathbb{C}} = T_1^{\mathbb{C}} \circ T_2^{\mathbb{C}} \quad \text{and} \quad \operatorname{tr}_V(T) = \operatorname{tr}_{V^{\mathbb{C}}}(T^{\mathbb{C}}).$$

Moreover, if $A \in \operatorname{End}(V^{\mathbb{C}})$, then

$$\operatorname{tr}_{(V^{\mathbb{C}})^{\mathbb{R}}} A = 2\operatorname{Re}\big(\operatorname{tr}_{V^{\mathbb{C}}} A\big).$$

Definition: Real and Compact Form

- If $\mathfrak{h}$ is a complex Lie algebra, a **real form** of $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}$.
- If $\mathfrak{g}$ is semisimple and $B_{\mathfrak{g}}$ is negative definite, then $\mathfrak{g}$ is called a **compact form** of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}$. By abuse of notation, if $\mathfrak{h}$ is a real Lie algebra, by a compact form we mean a compact form of $\mathfrak{h}^{\mathbb{C}}$.

Theorem II.45: [Hel01, Theorem III.6.3]

Every semisimple Lie algebra has a compact form.

Lemma II.46

Let $\mathfrak{g}_0$ be a real Lie algebra and $\mathfrak{g} := \mathfrak{g}_0^{\mathbb{C}}$ its complexification. Then

(1) $B_{\mathfrak{g}_0}(X, Y) = B_{\mathfrak{g}}(X, Y)$ *for all* $X, Y \in \mathfrak{g}$.

(2) $B_{\mathfrak{g}^{\mathbb{R}}}(X, Y) = 2\,\mathrm{Re}(B_{\mathfrak{g}}(X, Y)$ *for all* $X, Y \in \mathfrak{g}$.

(3) $\mathfrak{g}_0$ *semisimple* $\iff$ $\mathfrak{g}$ *semisimple* $\iff$ $\mathfrak{g}^{\mathbb{R}}$ *semisimple.*

This lemma follows from the fact that the map

$$\begin{aligned} e\colon \mathfrak{gl}(\mathfrak{g}_0) &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ T &\mapsto T^{\mathbb{C}} \end{aligned}$$

is a homomorphism of real Lie algebras and that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}_0 & \xrightarrow{\mathrm{ad}_{\mathfrak{g}_0}} & \mathfrak{gl}(\mathfrak{g}_0) \\ \big\downarrow & & \big\downarrow{\scriptstyle e} \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}_{\mathfrak{g}}} & \mathfrak{gl}(\mathfrak{g}) \end{array}$$

Back to orthogonal symmetric Lie algebras. Let $(\mathfrak{g}_0, \Theta_0)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{g} := \mathfrak{g}_0^{\mathbb{C}}$, $\Theta := \Theta_0^{\mathbb{C}}$ be the complexifications, and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ the complex conjugation. Then

$$\mathfrak{k},\ i\mathfrak{k},\ \mathfrak{p},\ i\mathfrak{p}$$

are $\mathbb{R}$-subspaces of $\mathfrak{g}^{\mathbb{R}}$ with the following bracket relations:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, i\mathfrak{p}] \subset i\mathfrak{p} \qquad \text{and} \qquad [i\mathfrak{p}, i\mathfrak{p}] = [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Thus $\mathfrak{g}^* := \mathfrak{k} + i\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{R}}$ with bracket coming from $\mathfrak{g}$

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([X, T] + [Y, Z])\,.$$

The conjugation $\tau\colon \mathfrak{g} \to \mathfrak{g}$ restricts to an involution $\Theta^* := \tau\big|_{\mathfrak{g}^*} \in \mathrm{End}(\mathfrak{g}^*)$.

Definition: Isomorphic orthogonal Symmetric Lie Algebras

Two orthogonal symmetric Lie algebras $(\mathfrak{g}_1, \Theta_1)$ and $(\mathfrak{g}_2, \Theta_2)$ are **isomorphic** $(\mathfrak{g}_1, \Theta_1) \cong (\mathfrak{g}_2, \Theta_2)$ if there exists a Lie algebra isomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\Theta_2 \circ \varphi = \varphi \circ \Theta_1$$

Proposition II.47

Let $(\mathfrak{g}_0, \Theta_0)$ be an orthogonal symmetric Lie algebra with $\mathfrak{g}_0$ semisimple. Then

(1) The pair $(\mathfrak{g}^*, \Theta^*)$ with $\Theta^* := \tau\big|_{\mathfrak{g}^*}$ is an orthogonal symmetric Lie algebra,

(2) $(\mathfrak{g}^*)^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}}$, $(\Theta^*)^{\mathbb{C}} = \Theta_0^{\mathbb{C}}$,

(3) $(\mathfrak{g}^*, \Theta^*)$ is effective if and only if $(\mathfrak{g}_0, \Theta_0)$ is effective,

(4) $(\mathfrak{g}^*, \Theta^*)$ is reduced if and only if $(\mathfrak{g}_0, \Theta_0)$ is reduced,

(5) The pair $(\mathfrak{g}_0, \Theta_0)$ is of non-compact type (resp. compact type) if and only if $(\mathfrak{g}^*, \Theta^*)$ is of compact type (resp. non-compact type),

(6) $(\mathfrak{g}_1, \Theta_1) \cong (\mathfrak{g}_2, \Theta_2)$ if and only if $(\mathfrak{g}_1^*, \Theta_1^*) \cong (\mathfrak{g}_2^*, \Theta_2^*)$ and

(7) $((\mathfrak{g}^*)^*, (\Theta^*)^*) = (\mathfrak{g}_0, \Theta_0)$.

Sketch of the proof. (1) Since $\tau \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$, $\mathfrak{k}$ is the part of the algebra fixed by $\Theta^* \in \mathrm{Aut}(\mathfrak{g}^*)$. So we only need to show that $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}^*$. Consider the $\mathbb{R}$-vector space isomorphism

$$\begin{aligned} \phi\colon \mathfrak{g}_0 &\to \mathfrak{g}^* \\ X + Y &\mapsto X + iY \end{aligned}$$

which induces the Lie group isomorphism

$$\begin{aligned} \Phi\colon \mathrm{GL}(\mathfrak{g}_0) &\to \mathrm{GL}(\mathfrak{g}^*) \\ A &\mapsto \phi A \phi^{-1}. \end{aligned}$$

Notice that for all $Z \in \mathfrak{k}$ we have

$$\phi \circ \mathrm{ad}_{\mathfrak{g}_0} Z = \mathrm{ad}_{\mathfrak{g}^*} Z \circ \phi.$$

Since $\mathfrak{k} < \mathfrak{g}_0$ is compactly embedded, there exists $U < GL(\mathfrak{g}_0)$ compact and connected such that $\mathrm{Lie}(U) = \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{k})$. Using Φ and its derivative

$$\begin{aligned} \mathrm{d}_{Id}\Phi\colon \mathfrak{gl}(\mathfrak{g}_0) &\to \mathfrak{gl}(\mathfrak{g}^*) \\ B &\mapsto \phi B \phi^{-1} \end{aligned}$$

we get $\mathrm{d}_{Id}\Phi\big(\mathrm{ad}_{\mathfrak{g}_0}\mathfrak{k}\big) = \mathrm{ad}_{\mathfrak{g}^*}\mathfrak{k}$, hence $\mathrm{ad}_{\mathfrak{g}^*}(\mathfrak{k}) = \mathrm{Lie}\big(\Phi(U)\big)$ and $\Phi(U) < GL(\mathfrak{g}^*)$ is a compact connected Lie group.

(2) The verfication is left to the reader.

(3) This follows from $\mathfrak{z}(\mathfrak{g}_0) \cap \mathfrak{k} = \mathfrak{z}(\mathfrak{g}^*) \cap \mathfrak{k}$.

(4) We recall that $(\mathfrak{g}_0, \Theta_0)$ is reduced if and only if any ideal $\mathfrak{n} < \mathfrak{g}_0$ that is contained in $\mathfrak{k}$ is trivial. Moreover, an ideal $\mathfrak{n} < \mathfrak{g}_0$ is contained in $\mathfrak{k}$ if and only if $\mathfrak{n} < \mathfrak{k}$ is an ideal and if $[\mathfrak{n}, \mathfrak{p}] = 0$.

Now $\mathfrak{n} < \mathfrak{k}$ is an ideal with $[\mathfrak{n}, \mathfrak{p}] = 0$ if and only if $[\mathfrak{n}, i\mathfrak{p}] = 0$, so $\mathfrak{n} < \mathfrak{g}_0$ is contained in $\mathfrak{k}$ if and only if $\mathfrak{n} < \mathfrak{g}^*$ is contained in $\mathfrak{k}$. Hence $(\mathfrak{g}_0, \Theta_0)$ is reduced if and only if $(\mathfrak{g}^*, \Theta^*)$ is reduced.

(5) Since $(\mathfrak{g}_0, \Theta_0)$ is effective semisimple, so is $(\mathfrak{g}^*, \Theta^*)$ according to Lemma II.46 and (3) above. Lemma II.46 further implies that for any $X, Y \in \mathfrak{p}$

$$\begin{aligned} B_{\mathfrak{g}_0}(X, Y) &= B_{\mathfrak{g}}(X, Y) \\ &= -B_{\mathfrak{g}}(iX, iY) \\ &= -B_{\mathfrak{g}^*}(iX, iY) \end{aligned}$$

since $(\mathfrak{g}^*)^{\mathbb{C}} \cong \mathfrak{g} := (\mathfrak{g}_0)^{\mathbb{C}}$.

(6) Any isomorphism of real Lie algebras extends to an isomorphism of the complexifications.

(7) Obviously we have $(\mathfrak{g}^*)^* = \mathfrak{g}_0$. The proof of $(\Theta^*)^* = \Theta_0$ is left to the reader. ■

Definition: Dual

$(\mathfrak{g}^*, \Theta^*)$ is called the **dual** of $(\mathfrak{g}_0, \Theta_0)$.

Example. $(\mathfrak{sl}(n, \mathbb{R}), \Theta)$ with $\Theta(X) = -X^t$. We saw already that

$$\mathfrak{sl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C}).$$

Now we claim that the dual of $(\mathfrak{sl}(n, \mathbb{R}), \Theta)$ is $(\mathfrak{su}(n, \mathbb{C}), \Theta^*)$.

In fact, if $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{sl}(n, \mathbb{R}) : X + X^t = 0\} \\ \mathfrak{p} &= \{X \in \mathfrak{sl}(n, \mathbb{R}) : X = X^t\}, \end{aligned}$$

then

$$\begin{aligned}\mathfrak{g}^* &= \mathfrak{k} + i\mathfrak{p} \\ &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) : Z = X + iY \text{ with } X + X^t = 0 \text{ and } Y = Y^t\} \\ &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) : Z + Z^* = 0\} \\ &= \mathfrak{su}(n,\mathbb{C})\end{aligned}$$

The corresponding symmetric spaces are

$$M = {}^{\mathrm{SL}(n,\mathbb{R})}\!\diagup_{\mathrm{SO}(n)}, \qquad M^* = {}^{\mathrm{SU}(n)}\!\diagup_{\mathrm{SO}(n)}$$

where M^* is compact.

Example. Let $\mathfrak{g} = \mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : X + X^t = 0\}$ and let $p, q \in \mathbb{N} \cup \{0\}$ be such that $p + q = n$. Define Θ_{pq} as

$$\begin{aligned}\Theta_{pq} \colon \mathfrak{gl}(p+q,\mathbb{C}) &\to \mathfrak{gl}(p+q,\mathbb{C}) \\ X &\mapsto I_{pq} X I_{pq}\end{aligned}$$

where $I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$.

It is easy to check that $\Theta_{pq}(\mathfrak{g}) = \mathfrak{g}$ and that $\Theta_{pq}^2 = Id$. We write $\mathfrak{g}$ in block form

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} : A + A^t = 0,\ D + D^t = 0,\ B \in M_{p,q}(\mathbb{R}) \right\}$$

and note that in this form

$$\Theta(X) = \Theta \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} = \begin{pmatrix} A & -B \\ B^t & D \end{pmatrix}.$$

It is then easy to see that the Cartan decomposition is given by

$$\begin{aligned}\mathfrak{k} &= \left\{ X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{so}(p+q) : A \in \mathfrak{so}(p), D \in \mathfrak{so}(q) \right\} \\ \mathfrak{p} &= \left\{ X = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \in \mathfrak{so}(p+q) : B \in M_{p,q}(\mathbb{R}) \right\}.\end{aligned}$$

Now we observe that

$$\begin{aligned}\mathfrak{g}^* &= \mathfrak{k} + i\mathfrak{p} \\ &= \left\{ \begin{pmatrix} A & iB \\ iB^t & D \end{pmatrix} : A + A^t = 0 = D + D^t,\ B \in M_{p,q}(\mathbb{R}) \right\}\end{aligned}$$

and define

$$\begin{aligned}\sigma\colon \mathfrak{gl}(n,\mathbb{C}) &\to \mathfrak{gl}(n,\mathbb{C})\\ Y &\mapsto \begin{pmatrix}-iI_p & 0\\ 0 & I_q\end{pmatrix} Y \begin{pmatrix}-iI_p & 0\\ 0 & I_q\end{pmatrix}\end{aligned}$$

such that

$$\sigma\begin{pmatrix}A & iB\\ iB^t & D\end{pmatrix} = \begin{pmatrix}A & B\\ B^t & D\end{pmatrix}.$$

This shows that σ is an isomorphism $\mathfrak{g}^* \xrightarrow{\cong} \mathfrak{so}(p,q)$ where

$$\mathfrak{so}(p,q) = \left\{\begin{pmatrix}A & B\\ B^t & D\end{pmatrix} : A + A^t = 0 = D + D^t,\ B \in M_{p,q}(\mathbb{R})\right\}.$$

Finally, the involution is given by $\Theta^* = \tau\big|_{\mathfrak{g}^*}$ with $\tau(X+iY) = X - iY$, that is

$$\Theta^*\begin{pmatrix}A & B\\ B^t & D\end{pmatrix} = \begin{pmatrix}A & -B\\ -B^t & D\end{pmatrix}.$$

In conclusion, $(\mathfrak{so}(p+q,\mathbb{R}),\Theta)$ and $(\mathfrak{so}(p,q),\Theta^*)$ are dual orthogonal symmetric Lie algebras. The corresponding Riemannian symmetric spaces are

$$M = {\mathrm{SO}(p+q,\mathbb{R})}\big/{\mathrm{SO}(p)\times\mathrm{SO}(q)}, \qquad M^* = {\mathrm{SO}(p,q)}\big/{\mathrm{SO}(p)\times\mathrm{SO}(q)}$$

where M is compact and M^* is not.

We will next show how this duality can be realised at the level of Riemannian symmetric pairs. To this end we will use the construction in Theorem II.41. Let $(\mathfrak{g}_0,\Theta_0)$ be a reduced semisimple orthogonal symmetric Lie algebra, $\mathfrak{g} := (\mathfrak{g}_0)^{\mathbb{C}}$ and $\tau\colon \mathfrak{g}\to\mathfrak{g}$ the complex conjugation with respect to $\mathfrak{g}_0$. Recall that $\mathfrak{g}^* = \mathfrak{k}\oplus i\mathfrak{p}$ and $\Theta^* = \tau\big|_{\mathfrak{g}^*}$. Then $(\mathfrak{g}^*)^{\mathbb{C}} = \mathfrak{g}$ and $(\Theta^*)^{\mathbb{C}} = \Theta_0^{\mathbb{C}}$.

Consider

$$\begin{aligned}e_0\colon\ \mathrm{Aut}(\mathfrak{g}_0) &\to \mathrm{Aut}(\mathfrak{g})\\ \alpha &\mapsto \alpha^{\mathbb{C}}\end{aligned}$$

which is an injective Lie group morphism satisfying

$$e_0(\Theta_0\circ\alpha\circ\Theta_0^{-1}) = (\Theta_0)^{\mathbb{C}}\circ\alpha^{\mathbb{C}}\circ(\Theta_0^{\mathbb{C}})^{-1}.$$

Denote σ_0 the restriction of the conjugation by $\Theta_0^{\mathbb{C}}$ to $e_0\big(\mathrm{Aut}(\mathfrak{g}_0)\big) < \mathrm{Aut}(\mathfrak{g})$. Analogously we consider

$$\begin{aligned}e^*\colon\ \mathrm{Aut}(\mathfrak{g}^*) &\to \mathrm{Aut}(\mathfrak{g})\\ \beta &\mapsto \beta^{\mathbb{C}}\end{aligned}$$

which is an injective Lie group morphism satisfying

$$e^*(\Theta^* \circ \beta \circ (\Theta^*)^{-1}) = (\Theta^*)^{\mathbb{C}} \circ \beta^{\mathbb{C}} \circ \big((\Theta^*)^{\mathbb{C}}\big)^{-1},$$

and denote σ^* the restriction of the conjugation by $(\Theta^*)^{\mathbb{C}}$ to $e^*\big(\mathrm{Aut}(\mathfrak{g}_0)\big) < \mathrm{Aut}(\mathfrak{g})$.

Proposition II.48

The groups $G_0 = e_0\big(\mathrm{Aut}(\mathfrak{g}_0)^\circ\big)$ and $G^* = e^*\big(\mathrm{Aut}(\mathfrak{g}^*)^\circ\big)$ are closed connected semisimple. Moreover, σ_0 defines an involution on G_0 and σ^* an involution on G^*. The group $K \coloneqq G_0 \cap G^*$ is compact and satisfies

$$(G_0^{\sigma_0})^\circ \subset K \subset G_0^{\sigma_0}, \qquad \big((G^*)^{\sigma^*}\big)^\circ \subset K \subset (G^*)^{\sigma^*}.$$

Moreover, the orthogonal symmetric Lie algebra associated to (G_0, K) is isomorphic to $(\mathfrak{g}_0, \Theta_0)$ and the orthogonal symmetric Lie algebra associated to (G^*, K) is isomorphic to $(\mathfrak{g}^*, \Theta^*)$.

Proof. It will turn out to be essential to understand the relation between τ, $\Theta_0^{\mathbb{C}}$, $(\Theta^*)^{\mathbb{C}}$ and τ^*, where τ^* denotes the complex conjugation of $\mathfrak{g} = \mathfrak{g}^* + i\mathfrak{g}^*$ with respect to $\mathfrak{g}^*$. All of these four maps are automorphisms of $\mathfrak{g}^{\mathbb{R}}$, that is $\mathfrak{g}$ seen as a real Lie algebra. It will be convenient to present the action of these automorphisms in a table:

	$\mathfrak{k}$	$i\mathfrak{k}$	$\mathfrak{p}$	$i\mathfrak{p}$
$\Theta_0^{\mathbb{C}}$	Id	Id	$-Id$	$-Id$
$(\Theta^*)^{\mathbb{C}}$	Id	Id	$-Id$	$-Id$
τ	Id	$-Id$	Id	$-Id$
τ^*	Id	$-Id$	$-Id$	Id

All these automorphisms have order two, commute pairwise and satisfy

$$\Theta_0^{\mathbb{C}} \circ \tau = \tau^*, \qquad (\Theta^*)^{\mathbb{C}} \circ \tau^* = \tau.$$

Hence if $\langle \alpha, \beta \rangle$ denotes the subgroup of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ generated by two elements α, β we have

$$\langle \Theta_0^{\mathbb{C}}, \tau \rangle = \langle (\Theta^*)^{\mathbb{C}}, \tau^* \rangle = \langle \tau, \tau^* \rangle.$$

Let us now define the following automorphisms of the Lie group $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$: For $\alpha \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ set

$$t(\alpha) = \tau\alpha\tau^{-1},\ t^*(\alpha) = \tau^*\alpha(\tau^*)^{-1},\ r(\alpha) = \Theta_0^{\mathbb{C}}\alpha(\Theta_0^{\mathbb{C}})^{-1} = (\Theta^*)^{\mathbb{C}}\alpha\big((\Theta^*)^{\mathbb{C}}\big)^{-1}.$$

Hence the set

$$\mathrm{Aut}(\mathfrak{g}) = \{\alpha \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{R}}) : \alpha(iZ) = i\alpha(Z)\ \forall\ Z \in \mathfrak{g}\}$$

is a closed subgroup of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ which is invariant by t: Indeed, for all $Z \in \mathfrak{g}$ we have

$$\tau(iZ) = -i\tau(Z)$$

and hence if $\alpha \in \mathrm{Aut}(\mathfrak{g})$

$$(\tau\alpha\tau^{-1})(iZ) = (\tau\alpha)\big(-i\tau^{-1}(Z)\big) = \tau\big(-i\alpha\tau^{-1}(Z)\big) = i\tau\alpha\tau^{-1}(Z).$$

We further claim that the image of $e_0\colon \mathrm{Aut}(\mathfrak{g}_0) \to \mathrm{Aut}(\mathfrak{g})$ coincides with $\big(\mathrm{Aut}\,\mathfrak{g}\big)^t$, the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ fixed by t. The proof is left as an easy verification. Thus the image of e_0 is closed and hence

$$e_0\colon \mathrm{Aut}\,\mathfrak{g}_0 \to \big(\mathrm{Aut}\,\mathfrak{g}\big)^t$$

is a Lie group isomorphism which implies that

$$G_0 = e_0\big((\mathrm{Aut}\,\mathfrak{g}_0)^\circ\big) = \big((\mathrm{Aut}\,\mathfrak{g})^t\big)^\circ$$

is closed connected semisimple.

The same argument applies to G^*.

Now we have the following relations in $\mathrm{Aut}(\mathrm{Aut}\,\mathfrak{g}^*)$: t, t^*,r are of order two and commute pairwise,

$$rt = t^*, \quad rt^* = t$$

and

$$\langle r, t\rangle = \langle r, t^*\rangle = \langle t, t^*\rangle$$

as subgroups of $\mathrm{Aut}(\mathrm{Aut}\,\mathfrak{g}^*)$.

Recall that $\sigma_0 = r\big|_{(\mathrm{Aut}\,\mathfrak{g})^t}$, $\sigma^* = r\big|_{(\mathrm{Aut}\,\mathfrak{g})^{t^*}}$. Moreover,

$$G_0 \cap G^* = \big((\mathrm{Aut}\,\mathfrak{g})^t\big)^\circ \cap \big((\mathrm{Aut}\,\mathfrak{g})^{t^*}\big)^\circ$$

is open in

$$(\mathrm{Aut}\,\mathfrak{g})^t \cap (\mathrm{Aut}\,\mathfrak{g})^{t^*} = (\mathrm{Aut}\,\mathfrak{g})^{\langle t,t^*\rangle} = (\mathrm{Aut}\,\mathfrak{g})^{\langle t,r\rangle} = (\mathrm{Aut}\,\mathfrak{g})^{\langle t^*,r\rangle}.$$

Thus $G_0 \cap G^*$ is open in $\big((\mathrm{Aut}\,\mathfrak{g})^t\big)^r$, but it is also contained in $G_0 = \big((\mathrm{Aut}\,\mathfrak{g})^t\big)^\circ$, hence it is open in

$$\big((\mathrm{Aut}\,\mathfrak{g})^t\big)^\circ \cap \big((\mathrm{Aut}\,\mathfrak{g})^t\big)^r = G_0^\sigma.$$

We conclude

$$\big(G_0^\sigma\big)^\circ \subset G_0 \cap G^* \subset G_0^\sigma.$$

The compactness then follows from the compactness of G_0^σ which is a direct consequence of Marcs Theorem IV.15. ∎

Definition: Compact dual

Let $(\mathfrak{g}_0, \Theta_0)$ be a reduced orthogonal symmetric Lie algebra of non-compact type and (G_0, K), (G^*, K) as above. Then we call $M^* = G^*/K$ the compact dual of the symmetric space $M = G_0/K$ of non-compact type. Observe here that K is connected and $G_0 = \mathrm{Iso}(M)^\circ$, $G^* = \mathrm{Iso}(M^*)^\circ$.

The following remarkable Theorem is the starting point of many interesting developments.

Theorem II.49

Let $M = G_0/K$ be a Riemannian symmetric space of non-compact type with compact dual $M^* = G^*/K$. Then there is a canonical isomorphism

$$\Omega^k(M)^{G_0} \cong H^k(M^*, \mathbb{R})$$

where

- *$\Omega^k(M)^{G_0}$ is the space of G_0-invariant smooth differential k-forms on M and*
- *$H^k(M^*, \mathbb{R})$ is the singular cohomology of M^* with $\mathbb{R}$-coefficients.*

Before we give a proof of this theorem we will need some notation and a few lemmata.

Notation. If V is a real vector space, we write $\mathrm{Alt}_k(V)$ for the space of alternating forms $V^k \to \mathbb{R}$ in k variables.

Lemma II.50

Let M be a Riemannian symmetric space, $G = \mathrm{Iso}(M)^\circ$, $o \in M$ and $K = \mathrm{Stab}_G(o)$. As $\pi\colon G \to G/K$ is the projection, $\mathrm{d}_e\pi\colon \mathfrak{p} \to T_oM$ is an isomorphism of vector spaces that commutes with the action of $\mathfrak{k}$ on $\mathfrak{p}$ via $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ and on T_oM via the differential of left translation. Then

$$\Omega^k(M)^G \to Alt_k(\mathfrak{p})^{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})}$$

is an isomorphism.

Sketch of Proof. The isomorphism is obtained by restricting $\omega \in \Omega^k(M)^G$ to $o \in M$, $\omega_0 \in \mathrm{Alt}_k(T_oM)$, and pulling it back: $(\mathrm{d}_e\pi)^*(\omega_o) \in \mathrm{Alt}_k(\mathfrak{p})$. ∎

Lemma II.51: (Cartan)

Let M be a Riemannian symmetric space , $G = \mathrm{Iso}(M)^\circ$ and $\omega \in \Omega^k(M)^G$. Then

$$\mathrm{d}\omega = 0.$$

Proof. Take $o \in M$, $s_o \in G$ the geodesic symmetry at o and let $\omega \in \Omega(M)^G$. Since for all $g \in G$ we have $s_o g s_o \in G$ we get

$$\begin{aligned}\omega &= (s_o g s_o)^* \omega \\ &= s_o^* g^* s_o^* \omega,\end{aligned}$$

hence from $s_o^* = \left(s_o^*\right)^{-1}$

$$s_o^* \omega = g^* s_o^* \omega.$$

This shows that $s_o^*\omega \in \Omega^k(M)^G$. Moreover, $s_o\big|_{T_oM} = -Id$ and thus

$$(s_o^*\omega)_o = (-1)^k \omega_o$$

and by G-invariance

$$(s_o^*\omega)_x = (-1)^k \omega_x \qquad \text{for all} \quad x \in M.$$

So the forms $(-1)^k\omega$ and $s_o^*\omega$ coincide, and applying d we get

$$(-1)^k \mathrm{d}\omega = \mathrm{d}(s_o^*\omega) = s_o^*(\mathrm{d}\omega).$$

But since $\mathrm{d}\omega$ is an invariant $(k+1)$-form we have $s_o^*(\mathrm{d}\omega) = (-1)^{k+1}\mathrm{d}\omega$ which implies $\mathrm{d}\omega = -\mathrm{d}\omega$ and hence $\mathrm{d}\omega = 0$. ∎

Lemma II.52

Let U be a compact connected Lie group acting smoothly on a smooth manifold X. The inclusion of complexes

$$\Omega^k(X)^U \hookrightarrow \Omega^k(X)$$

induces an isomorphism in cohomology

$$\Omega^k(X)^U \cong H^k_{dR}(X).$$

Proof. As U is compact, there is a normalized Haar measure $\mathrm{d}\mu$ on U.

Injectivity: Let $\alpha \in \Omega^k(X)^U$ and assume that α is exact in $\Omega^k(X)$, that is $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^{k-1}(X)$. Since the U-action commutes with d we get

$$\begin{aligned}\alpha &= u^*\alpha \\ &= u^*\mathrm{d}\beta \\ &= \mathrm{d}(u^*\beta)\end{aligned}$$

for all $u \in U$, hence

$$\begin{aligned}\alpha &= \int_U u^*\alpha \mathrm{d}\mu(u) \\ &= \int_U \mathrm{d}(u^*\beta)\,\mathrm{d}\mu(u) \\ &= \mathrm{d}\underbrace{\left(\int_U u^*\beta \mathrm{d}\mu(u)\right)}_{\in\Omega^{k-1}(X)^U}.\end{aligned}$$

Surjectivity: Let $\alpha \in \Omega^k(U)$ such that $\mathrm{d}\alpha = 0$. As U is connecte, every $u \in U$ is diffeotopic to the identity $Id \in U$. Thus α and $u^*\alpha$ represent the same class in $H^*_{dR}(X)$, and for every C^1-cycle $z \in H_k(X;\mathbb{R})$ we have

$$\int_z \alpha = \int_z u^*\alpha \quad \text{for any} \quad u \in U.$$

Using Fubini we conclude

$$\begin{aligned}\int_z \alpha &= \int_U \left(\int_z (u^*\alpha)\right) \mathrm{d}\mu(u) \\ &= \int_z \int_U u^*\alpha\ \mathrm{d}\mu(u),\end{aligned}$$

which shows that

$$\int_z \left(\alpha - \int_U u^*\alpha\ \mathrm{d}\mu(u)\right) = 0 \qquad \text{for all} \quad z \in H_k(X;\mathbb{R}).$$

De Rham's Theorem now implies that α and $\int_U u^*\alpha\ \mathrm{d}\mu(u)$ represent the same cohomology class. In particular,

$$\alpha = \int_U u^*\alpha\ \mathrm{d}\mu(u)$$

is U-invariant which proves surjectivity. ■

Proof of Theorem II.49. Denote $G_0 = \mathrm{Iso}(M)^\circ$, $G^* = \mathrm{Iso}(M^*)^\circ$ and $o = eK \in M \cap M^*$. Then according to Lemma II.50

$$\Omega^k(M)^{G_0} \cong \mathrm{Alt}_k(\mathfrak{p})^{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})}$$

Now

$$\begin{aligned} \mathfrak{p} &\to i\mathfrak{p} \\ X &\mapsto iX \end{aligned}$$

is an $\mathrm{Ad}(K)$-equivariant isomorphism of real vector spaces, hence

$$\mathrm{Alt}_k(\mathfrak{p})^{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})} \cong \mathrm{Alt}_k(i\mathfrak{p})^{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})}$$

and the latter is isomorphic to $\Omega^k(M^*)^{G^*}$ again by Lemma II.50. Since G^* is a compact connected Lie group, the inclusion of complexes

$$\Omega^k(M^*)^{G^*} \to \Omega^k(M^*)$$

induces an isomorphism in cohomology $\Omega^k(M^*)^{G^*} \cong H^k_{dR}(M^*, \mathbb{R})$ according to Lemma II.52. But by Lemma II.51 $(\Omega^k(M^*)^{G^*}, \mathrm{d})$ is equal to its cohomology and hence

$$\Omega^k(M^*)^{G^*} \cong H^k_{dR}(M^*, \mathbb{R}),$$

the latter being De Rham cohomology which itself is isomorphic to $H^k(M^*, \mathbb{R})$ by De Rham's Theorem. ■

Chapter III

Symmetric Spaces of Non-Compact Type

III.1 Symmetric spaces are CAT(0)

Definition: Geodesic and Comparison Triangle

- A metric space (X, d_X) is called **geodesic** if for every two points $x, y \in X$ there is a continuous path

$$\gamma\colon [0, d_X(x,y)] \to X$$

from x to y and such that $l(\gamma) = d_X(x,y)$, where

$$\ell(\gamma) := \sup\left\{\sum_{j=0}^{n-1} d_X(\gamma(t_j), \gamma(t_{j+1}) : 0 = t_0 < \cdots < t_n = d_X(x,y)\right\}$$

and the supremum is taken over all partitions of $[0, d_X(x,y)]$.

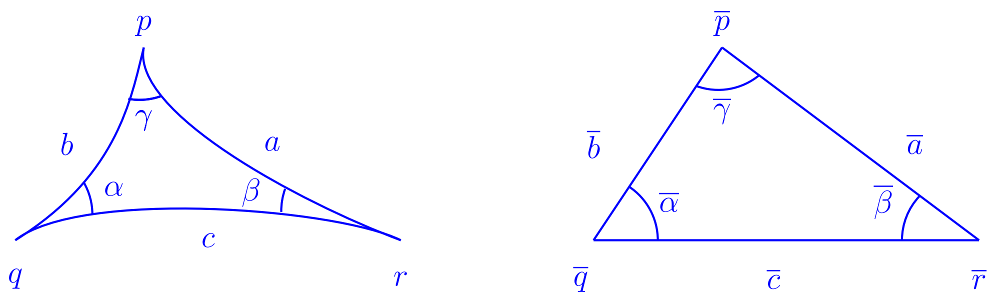
- A **geodesic triangle** $\Delta(p,q,r)$ in a geodesic metric space X consists of three points $p, q, r \in X$ and geodesic segments $[p,q]$, $[q,r]$ and $[p,r]$ that join them and whose lengths is the distance between the endpoints[a].

- Given $\Delta(p,q,r)$, a **comparison triangle** is a triangle $\overline{\Delta} = \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ in $\mathbb{E}^2$ whose sides are geodesic segments of the same length as the sides in Δ.

- Given a point $x \in [p,q]$, a **comparison point** for it is a point $\overline{x} \in [\overline{p}, \overline{q}]$ such that

$$d_X(p,x) = d_{\mathbb{E}^2}(\overline{p}, \overline{x}).$$

[a] The notation $\Delta(p,q,r)$, that is the lack of indication of the geodesic segments, will soon be justified (see the Remark before Proposition III.1.



It is easy to see that comparison triangles always exist and are unique up to isometries.

Definition: CAT(0)

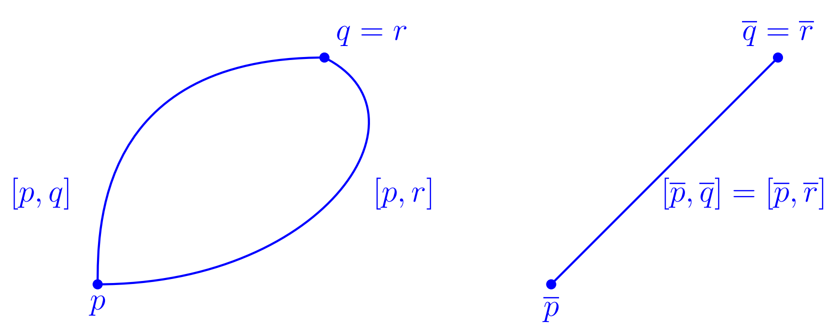
A geodesic metric space is **CAT**(0) if for every geodesic triangle $\Delta(p, q, r)$ with comparison triangle $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ and points $x, y \in \Delta(p, q, r)$ with comparison points $\overline{x}, \overline{y} \in \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ the following inequality holds

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}) .$$

Remark. It is intuitively obvious that in a CAT(0)-space triangles are **thin**, that is, $\alpha \leq \overline{\alpha}$, $\beta \leq \overline{\beta}$ and $\gamma \leq \overline{\gamma}$. (This will be proven in Corollary **??**(2).)

More generally, in a CAT(κ) space triangles are thinner than triangles in a model space of constant curvature κ. For CAT(-1) the hyperbolic space is used, for CAT(0) the euclidean plane and for CAT(1) we use the sphere.

Remark. CAT(0)-spaces are **uniquely geodesic**. To see this, let us take a geodesic triangle $\Delta(p, q, r) \subset X$ with comparison triangle $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r}) \subset \mathbb{E}^2$. Let $X \in [p, q]$ and $y \in [p, r]$ be such that $d_X(p, x) = d_X(p, y)$. If now $q = r$, so that $\overline{p} = \overline{r}$, the geodesic sides $[\overline{p}, \overline{q}]$ and $[\overline{p}, \overline{r}]$ must coincide since $\mathbb{E}^2$ is uniquely geodesic and hence $\overline{x} = \overline{y}$.



Since

$$d_X(x, x') \leq d_{\mathbb{E}^2}(\overline{x}, \overline{x'}) = 0,$$

then $x = y$ and hence $[p, q] = [p, r]$.

Notice that we will show in Theorem III.2 that Riemannian symmetric spaces of non-compact type are CAT(0). However the proof of this fact uses that they are uniquely geodesic, which will be shown in Proposition **??**.

Proposition III.1

Let (X, d) be a complete CAT(0)-space.

(1) If $S \subset X$ is a bounded set and

$$r_x := \inf\left\{r > 0 : S \subset \overline{B}(x, r) \text{ for some } x \in X\right\}$$

then there exists a unique $x_s \in X$ such that $S \subset \overline{B}(x_s, r_s)$. We call this the ***circumcenter*** *of S.*

(2) Let $C \subset X$ be a closed convex set. Then there exists a unique $p_C(x) \in C$ such that

$$d_X(x, p_C(x)) \leq d(x, C) := \inf\{d_X(x, y) : y \in C\}$$

(3) Let $\gamma_1, \gamma_2 \colon \mathbb{R} \to X$ be two geodesics parametrised by arclength. The map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\mapsto d_X(\gamma_1(t), \gamma_2(t)) \end{aligned}$$

is convex.

Proof. (1) Let $(r_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $r_n \to r_s$ that has the property that there is $x_n \in X$ such that $S \subset \overline{B}(x_n, r_n)$.

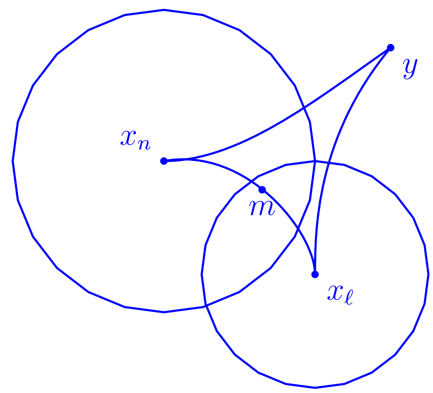
Claim: $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

If so, $(x_n)_{n\in\mathbb{N}}$ converges since X is complete. We set $x_s := \lim x_n$, so that $S \subset \overline{B}(x_s, r_s)$. To see that it is unique, let x_s, x'_s be two such points and let us define

$$y_n = \begin{cases} x_s & n \text{ even} \\ x'_s & n \text{ odd.} \end{cases}$$

Thus $S \subset (\overline{B}(x_s, r_s) \cap \overline{B}(x'_s, r_s))$, so that $S \subset \overline{B}(y_n, r_s)$. Again by the claim that the sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy, hence convergent, it follows that $x_s = x'_s$.

Proof of claim: Let $\varepsilon > 0$. Take then two points x_n, x_ℓ of the sequence and define m to be the midpoint of the geodesic $[x_n, x_\ell]$ which connects them.

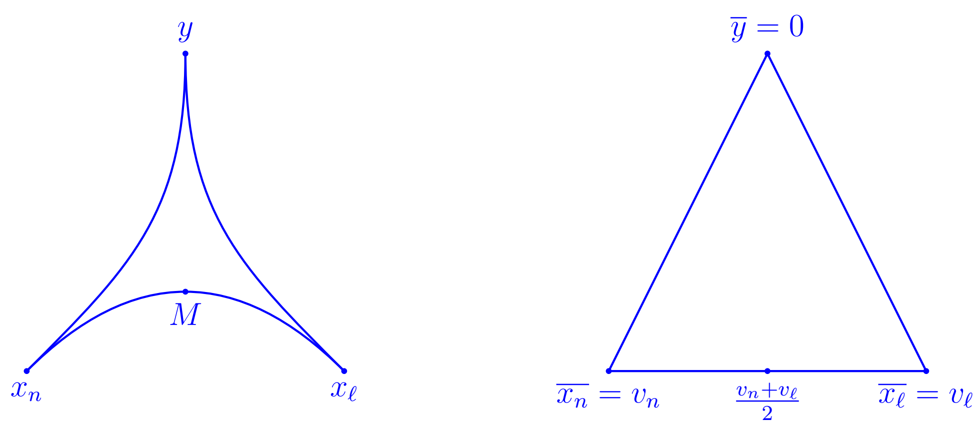


Then by definition of r_s there is a $y \in S$ such that

$$d_X(y,m)^2 > r_s^2 - \varepsilon.$$

If not, then for every $y \in S$ we had $d_X(y,m)^2 \leq r_s^2 - \varepsilon$ which contradicts the definition of r_s as it would imply $S \subset \overline{B}(m, r_s - \varepsilon)$.

Consider then the geodesic triangle $\Delta(x_n, x_\ell, y)$ and its comparison triangle $\overline{\Delta}(\overline{x_n}, \overline{x_\ell}, \overline{y})$.



Write $\overline{y} = 0$, $\overline{x_n} = v_n$ and $\overline{x_\ell} = v_\ell$ and note that the comparison point of $M \in [x_n, x_\ell]$ i.e. $\overline{M} \in [\overline{x_n}, \overline{x_\ell}]$ corresponds to $\frac{v_n+v_\ell}{2}$. Then

$$\left\| \frac{v_n + v_\ell}{2} \right\|^2 = d_{\mathbb{E}^2}(\overline{y}, \overline{M})^2 \overset{\text{CAT}(0)}{\geq} d_X(y,m)^2 > r_s^2 - \varepsilon$$

implies that

$$-2\langle v_n, v_\ell \rangle < \|v_n\|^2 + \|v_\ell\|^2 - 4(r_s^2 - \varepsilon).$$

But then

$$\|v_n - v_\ell\|^2 = \|v_n\|^2 + \|v_\ell\|^2 - 2\langle v_n, v_\ell\rangle < 2\|v_n\|^2 + 2\|v_\ell\|^2 - 4(r_s^2 - \varepsilon)$$

and by definition of r_s, given the chosen $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $r_k^2 \leq r_s + \varepsilon$ for all $k \geq N$. Hence choose $n, m \geq N$ and note

$$\|v_n\|^2 = d_{\mathbb{E}^2}(\overline{x_n}, \overline{y}) < r_s^2 + \varepsilon \qquad \text{and} \qquad \|v_\ell\|^2 = d_{\mathbb{E}^2}(\overline{x_\ell}, \overline{y}) < r_s^2 + \varepsilon.$$

Thus it follows that

$$d_X(x_n, x_\ell) \leq d_{\mathbb{E}^2}(\overline{x_n}, \overline{x_\ell})^2 = \|v_n - v_\ell\|^2 \leq 8\varepsilon$$

and thereby that (x_n) is Cauchy.

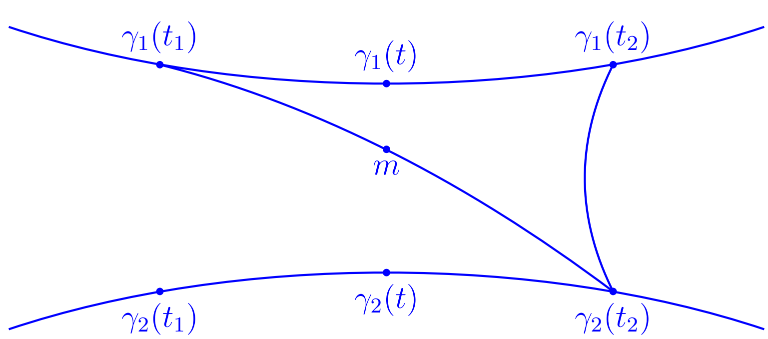
(2) Exercise.

(3) Write $f(t) \coloneqq d(\gamma_1(t), \gamma_2(t))$. For t_1, t_2 we note $t \coloneqq \frac{t_1+t_2}{2}$ and want to show that

$$f(t) \leq \frac{1}{2}(f(t_1) + f(t_2)).$$

Consider the midpoint m of the geodesic segment $[\gamma_1(t_1), \gamma_2(t_2)]$ and the comparison triangle $\overline{\Delta}(\overline{\gamma_1(t_1)}, \overline{\gamma_1(t_2)}, \overline{\gamma_2(t_2)})$. Since the two triangles $\overline{\Delta}(\overline{\gamma_1(t_1)}, \overline{m}, \overline{\gamma_1(t)})$ and $\overline{\Delta}(\overline{\gamma_1(t_1)}, \overline{\gamma_2(t_2)}, \overline{\gamma_1(t_2)})$ are similar and $\overline{\gamma_1(t)}$ is the midpoint of $[\overline{\gamma_1(t_1)}, \overline{\gamma_2(t_2)}]$, it follows that

$$d_{\mathbb{E}^2}(\overline{\gamma_1(t)}, \overline{m}) = \frac{1}{2} d_{\mathbb{E}^2}(\overline{\gamma_1(t_2)}, \overline{\gamma_2(t_2)})$$



Note that $\overline{\gamma_1(t)}$ is the midpoint of $[\overline{\gamma_1(t_1)}, \overline{\gamma_1(t_2)}]$ which implies

$$\begin{aligned} d_X(\gamma_1(t), m) &\leq d_{\mathbb{E}^2}(\overline{\gamma_1(t)}, \overline{m}) \\ &= \frac{1}{2} d_{\mathbb{E}^2}(\overline{\gamma_1(t_2)}, \overline{\gamma_2(t_2)}) \\ &= \frac{1}{2} d_X(\gamma_1(t_2), \gamma_2(t_2)). \end{aligned}$$

Likewise, by considering the geodesic triangle $\Delta(\gamma_1(t_1), \gamma_2(t_1), \gamma_2(t_2))$, we show that

$$d_X(\gamma_2(t), m) \leq \frac{1}{2} d_X(\gamma_1(t_1), \gamma_2(t_1)).$$

Finally we observe that

$$\begin{aligned} f(t) &= d_X(\gamma_1(t), \gamma_2(t)) \\ &= d_X(\gamma_1(t), m) + d_X(m, \gamma_2(t) \\ &\leq \frac{1}{2} \left(d_X(\gamma_1(t_2), \gamma_2(t_2)) + d_X(\gamma_1(t_1), \gamma_2(t_1)) \right) \\ &= \frac{1}{2}(f(t_2) + f(t_1)) \end{aligned}$$

■

Theorem III.2

Riemannian symmetric spaces of non-compact type are CAT(0).

As remarked already before of the remark right before Proposition **??**, we coudl infer that Riemannian symmetric spaces of non-compact type are uniquely geodesic. However the proof of Theorem III.2 uses Proposition **??**, where the fact that a Riemannian symmetric space is uniquely geodesic is proven directly.

Proposition III.3

Let M be a Riemannian symmetric space of non-compact type. Then $\operatorname{Exp}_o \colon T_oM \to M$ is a distance increasing diffeomorphism and hence M is uniquely geodesic.

Proof. We want to show that for $X \in T_oM$ and for $\xi \in T_X(T_oM) \cong T_o(M)$, if $X \in \mathfrak{p}$

$$\|\mathrm{d}_X \operatorname{Exp}_o(\xi)\| \geq \|\xi\| \, ..$$

According to Corollary II.24 $X \in \mathfrak{p}$

$$\mathrm{d}_X(\operatorname{Exp}_o \circ \mathrm{d}_e\pi) = (\mathrm{d}_o L_{\exp(X)} \circ \mathrm{d}_e\pi) \circ \left(\sum_{n=0}^{\infty} \frac{(T_X\big|_{\mathfrak{p}})^n}{(2n+1)!} \right)$$

where $T_X = (\mathrm{ad}_{\mathfrak{g}} X)^2$. Since both $\mathrm{d}_e\pi$ and $\mathrm{d}_o L_{\exp X}\colon T_oM \to T_{(\exp X)_* o}M$ preserve the scalar product (the first by definition and the second since it is the parallel transport along $t \mapsto \exp tX \cdot o$), the assertion can be true only if

$$T := \sum_{n=0}^{\infty} \frac{(T_X\big|_{\mathfrak{p}})^n}{(2n+1)!}$$

is distance increasing.

To see this, we first claim that $T_X|_{\mathfrak{p}}$ is diagonalizable. In fact, for $X, Y \in \mathfrak{p}$

$$\begin{aligned} B_{\mathfrak{g}}(T_X Y, Y) &= B_{\mathfrak{g}}((\mathrm{ad}_{\mathfrak{g}} X)^2 Y, Y) \\ &= -B_{\mathfrak{g}}(\mathrm{ad}_{\mathfrak{g}}(X)Y, \mathrm{ad}_{\mathfrak{g}}(X)Y) \\ &\geq 0 \end{aligned}$$

because $\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{p}) \subset \mathfrak{k}$ and $B_{\mathfrak{g}}|_{\mathfrak{k}\times\mathfrak{k}} \leq 0$. It follows that there exists an orthonormal basis $\{e_1, .., e_n\}$ of $\mathfrak{p}$ such that $T_X = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_j > 0$. Therefore

$$T = \mathrm{diag}(\mu_1, ..., \mu_n) \qquad \text{with} \qquad \mu_j = \sum_{k=0}^{\infty} \frac{\lambda_i^k}{(2k+1)!} \geq 1$$

and hence T is distance increasing. ■

Remark. The assertion of Proposition III.3 is actually true for all complete Riemannian manifolds of non-positive curvature.

Corollary III.4

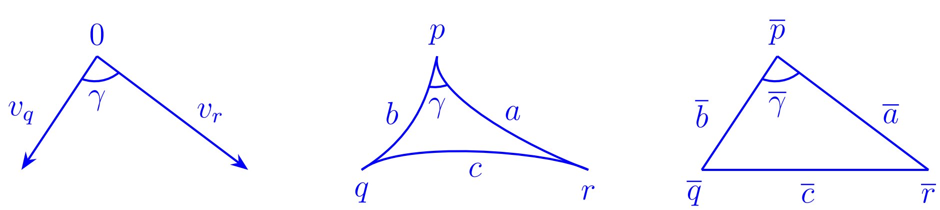
(1) (Law of cosine) Let M be a Riemannian symmetric space of non-compact type, Δ a geodesic triangle with a, b, c as sides and opposite angles α, β, γ. Then

$$c^2 \geq a^2 + b^2 - 2ab\cos(\gamma)$$

(2) If $\Delta = \Delta(p, q, r)$ with comparison triangle $\overline{\Delta} = \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$, then

$$\alpha \leq \overline{\alpha}, \qquad \beta \leq \overline{\beta} \qquad \text{and} \qquad \gamma \leq \overline{\gamma}.$$

Proof. Consider the geodesic segment $[p, q], [p, r]$ and $[q, r]$ in M. Then $[p, q]$ and $[p, r]$ are determined by tangent vectors $v_q, v_r \in T_pM$ such that $\|v_q\| = b$ and $\|v_r\| = a$, so that $q = \mathrm{Exp}_p(v_q)$ and $r = \mathrm{Exp}_p(v_r)$



By definition, the angle at 0 between v_q and v_r is γ, so that

$$\|v_r - w_r\|^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

By Proposition III.3

$$\begin{aligned} c^2 &= d_x(q,r)^2 \\ &= d_X(\mathrm{Exp}(v_q), \mathrm{Exp}(v_r))^2 \\ &\geq \|(v_q - v_r\|^2 \\ &= a^2 + b^2 - 2ab\cos(\gamma). \end{aligned}$$

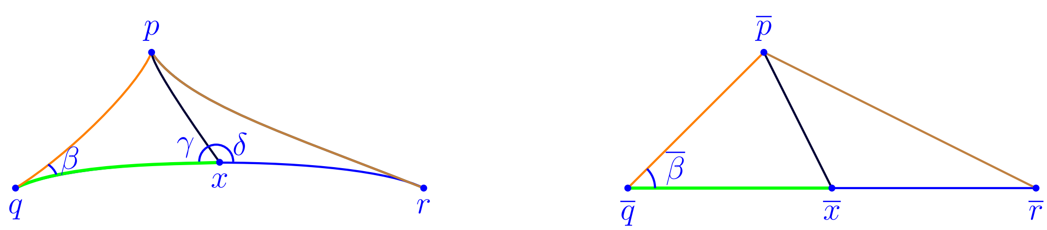
(2) From the first part we note that

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\overline{\gamma}) \\ c^2 &\geq a^2 + b^2 - 2ab\cos(\gamma) \end{aligned}$$

and thus $\gamma \leq \overline{\gamma}$. ■

Proof of Theorem III.2. We proceed in two steps. First we show that any space that satisfies the Law of Cosine in Corollary III.4 satisfies the CAT(0) inequality for a vertex and a point inside a geodesic side, then we take two generic points.

(1) Let $\Delta = \Delta(p,q,r)$v be a geodesic triangle and $\overline{\Delta} = \overline{\Delta}(,\overline{p},\overline{q})r$ be its comparison triangle. Take $x \in [q,r]$ with $\overline{x} \in [\overline{q},\overline{r}]$.

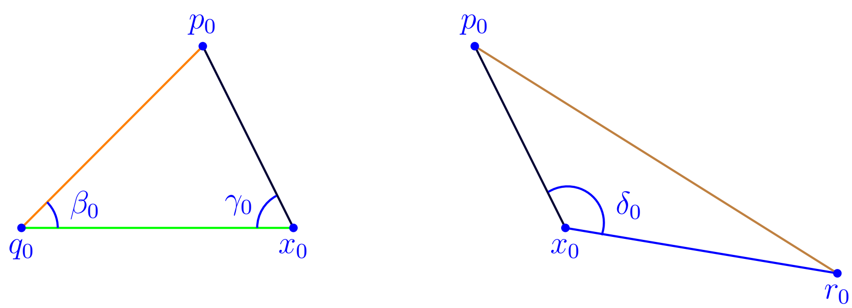


By definition of comparison triangle,

$$d_M(q,x) = d_{\mathbb{E}^2}(\overline{q},\overline{x}) \text{ and } d_M(x,r) = d_{\mathbb{E}^2}(\overline{x},\overline{r})\,,$$

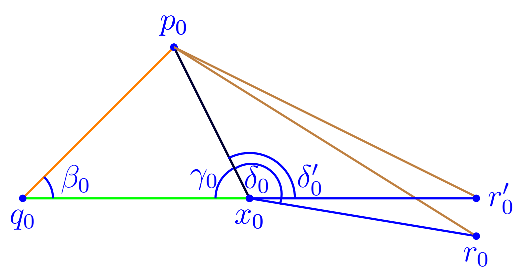
but we want to compare $d_M(x,r)$ and $d_{\mathbb{E}^2}(\overline{x},\overline{r})$. The idea is to consider x as the vertex of two geodesic triangles $\Delta(p,q,x)$ and $\Delta(x,r,p)$

$$d_M(p,x) \leq d_{\mathbb{E}^2}(\overline{p},\overline{x}).$$



It follows from Corollary III.4 that $\gamma \leq \gamma_0$ and $\delta \leq \delta_0$, so that $\pi = \gamma + \delta \leq \gamma_0 + \delta_0$. Thus q_0, x_0 and r_0 are not necessarily collinear. Let r_0' be collinear with q_0 and x_0 and such that

$$d_{\mathbb{E}^2}(x_0, r_0') = d_{\mathbb{E}^2}(x_0, r_0) = d_M(x, r)\,.$$



Then $\Delta_0' \leq \delta_0$, so that

$$d_{\mathbb{E}^2}(p_0, r_0') \leq d_{\mathbb{E}^2}(p_0, r_0) = d_M(p, r) = d_{\mathbb{E}^2}(\overline{p}, \overline{r})\,. \tag{III.1}$$

Applying Corollary III.4 to the triangle $\overline{\Delta}(\overline{p_0}, \overline{r_0'}, \overline{q_0})$ we obtain

$$d_{\mathbb{E}^2}(p_0, r_0')^2 = d_{\mathbb{E}^2}(p_0, q_0)^2 + d_{\mathbb{E}^2}(q_0, r_0')^2 - 2 d_{\mathbb{E}^2}(p_0, q_0) d_{\mathbb{E}^2}(q_0, r_0') \cos \beta_0$$

and to the triangle $\overline{\Delta}(\overline{p}, \overline{r}, \overline{q})$ we obtain

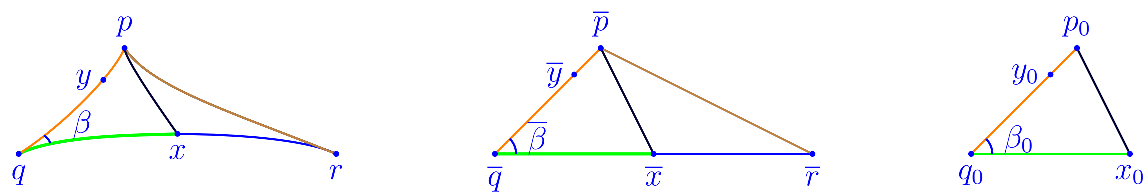
$$d_{\mathbb{E}^2}(\overline{p}, \overline{r})^2 = d_{\mathbb{E}^2}(\overline{p}, \overline{q})^2 + d_{\mathbb{E}^2}(\overline{q}, \overline{r})^2 - 2 d_{\mathbb{E}^2}(\overline{p}, \overline{q}) d_{\mathbb{E}^2}(\overline{q}, \overline{r}) \cos \overline{\beta}\,.$$

From (III.1) and since $d_{\mathbb{E}^2}(p_0, q_0) = d_{\mathbb{E}^2}((\overline{p}, \overline{q})$, and $d_{\mathbb{E}^2}(p_0, r_0') = d_{\mathbb{E}^2}(\overline{p}, \overline{r})$, we obtain that $\beta_0 \leq \overline{\beta}$. It follows then that

$$d_{\mathbb{E}^2}(\overline{p}, \overline{x}) \geq d_{\mathbb{E}^2}(p_0, q_0) = d_M(p, x)\,, \tag{III.2}$$

where the inequality comes from the fact that we are in Euclidean geometry and the equality from the comparison of $\Delta(q, x, p)$ with $\overline{\Delta}(q_0, x_0, p_0)$.

(2) Let now y be a generic point on one of the geodesic sides, for example $y \in [p, q]$.



From (1) applied to the geodesic triangle $\Delta(q, x, p)$ and to $\overline{\Delta}(x_0, p_0, q_0)$ we obtain that

$$d_M(x, y) \leq d_{\mathbb{E}^2}(x_0, y_0) .$$

Still from the proof of (1) we know that $\beta_) \leq \overline{\beta}$m which, by relabeling the vertex implies that

$$d_{\mathbb{E}^2}(x_0, y_0) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}) .$$

The two last formulas give the desired assertion.

■

Theorem III.5

Let M be a Riemannian symmetric space of non-compact type, $o \in M$ and $K = \mathrm{Stab}_G(o)$. Then

(1) Any compact subgroup $U < G$ has a fixed point in M. In particular $\mathrm{Stab}_G(p)$ is a maximal compact subgroup for every $p \in M$. All maximal subgroups arise in this way and they are all conjugate.

(2) The map from M to the subgroups in G defined as $p \mapsto \mathrm{Stab}_G(p)$ is injective.

Proof. (1) If U is compact, $S = U_*o$ is bounded and thus has a circumcenterx_S whose existence was proven in Proposition III.1. Since S is by construction U-invariant, the circumcenter is fixed by U. Thus $\mathrm{Stab}_G(p)$ is a maximal compact subgroup in G.

(2) By the first part, any compact subgroup $U < G$ is contained in some $\mathrm{Stab}_G(p)$. Since maximal compact subgroups are conjugate, they fix two points in M (by transitivity).

Also if $p \neq q$, then $\mathrm{Stab}_G(p) \neq \mathrm{Stab}_G(q)$. If not, there would exist $0 \neq Z \in \mathfrak{p}$ such that $\mathrm{ad}_{\mathfrak{g}}(K)Z = 0$. This would be true in one of the irreducible symmetric spaces. But then this would have dimension 1 which is impossible since M is of non-compact type. ■

Remark. If M is a Riemannian symmetric space of non-compact type, we can thus write it as

$$M = {}^{G}\!\diagup_{K}$$

where $K < G$ is a maximal compact subgroup. If M^* is of compact type, we can write

$$M^* = {}^{G^*}\!\diagup_{K}$$

where $K < G$ is again a maximal compact subgroup.

III.2 Flats and Rank

Definition

A k-**flat** F in a Riemannian symmetric space M is a totally geodesic submanifold isometric to $\mathbb{E}^k$ for some $k \in \mathbb{N}$, that is, for every $p \in M$ and any $X, Y \in T_pM$ orthonormal we have

$$\kappa_p(\mathrm{Span}\{X, Y\}) = 0.$$

Remark. Notice that a 1-flat is nothing but a geodesic. Moreover, if the sectional curvature of the symmetric space is strictly negative, then the symmetric space must be of rank one. The rank one symmetric spaces of non-compact type are exactly the real, complex and quaternionic hyperbolic spaces and the hyperbolic plane over the Cayley numbers.

Theorem III.6

(1) Let (G,K) be a Riemannian symmetric pair of compact or non-compact type, $\mathrm{d}_e\pi\colon \mathfrak{p} \to T_oM$ and $\mathrm{Exp}_o \circ \mathrm{d}_e\pi\colon \mathfrak{p} \to M$. Then $F \ni o$ is a flat subspace if and only if

$$F = (\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(\mathfrak{a})$$

where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subspace.

(2) If $M = G/K$ is of non-compact type, $\mathrm{Exp}_o \circ \mathrm{d}_e\pi\colon \mathfrak{a} \to F$ is an isometry.

Proof. (1) Let F be totally geodesic. Then $F = (\mathrm{Exp}_o \circ \mathrm{d}_e\pi)(\mathfrak{n})$ where $\mathfrak{a}$ is a Lie triple system. Thus F is a Riemannian symmetric space with Riemannian symmetric pair (G', K') where $\mathfrak{g}' = \mathfrak{n} \oplus [\mathfrak{n}, \mathfrak{n}]$ and $K' = K \cup G'$. Since G' acts transitively on F, it is enough to consider the sectional curvature at the basepoint $o \in M$. We then note that for all $X, Y \in \mathfrak{p}$ we have from (II.18) that $\kappa_o(\mathrm{Span}\{X,Y\}) = -B_{\mathfrak{g}}([X,Y],[X,Y])$, so that for all $X, Y \in \mathfrak{n}$,

$$\kappa_o(\mathrm{Span}\{X,Y\}) = 0 \qquad \Leftrightarrow \qquad [X,Y] = 0\,.$$

(2) For every $X \in \mathfrak{p}$ we note that

$$d_X(\mathrm{Exp}_o \circ \mathrm{d}_e\pi) = (\mathrm{d}_o L_{\exp X} \circ \mathrm{d}_e\pi) \circ \sum_{n=0}^{\infty} \frac{(T_X\big|_{\mathfrak{p}})^n}{(2n+1)!}$$

where the first term preserves the inner product and the second is in general distance increasing. However, if $X \in \mathfrak{a}$, then $T_X\big|_{\mathfrak{a}} = 0$. Since $\mathrm{Exp}_o \circ \mathrm{d}_e\pi$ is a diffeomorphism (Proposition III.1), it is a Riemannian isometry and thus an isometry. ■

We need to find a way to determine the maximal Abelian subalgebras in $\mathfrak{p}$. To this purpose, if $X \in \mathfrak{g}$ we write

$$\mathrm{Centr}_{\mathfrak{g}}(X) \coloneqq \{Y \in \mathfrak{g} : [X,Y] = 0\}.$$

If $\mathfrak{a} \subset \mathfrak{p}$ is Abelian and $X \in \mathfrak{a}$, then $\mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$. It follows that if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is Abelian, it must be maximal.

Definition: Regular and Singular Elements

$X \in \mathfrak{p}$ is called **regular** if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is Abelian and **singular** otherwise.

Theorem III.7: **[Hel01, Lemma V.6.3(i)]**

Let $M = G/K$ be a Riemannian symmetric space of compact or non-compact type, $G = \mathrm{Iso}(M)^\circ$ and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subspace. Then there exists an element $X \in \mathfrak{a}$ such that $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$.

Proof. If $\mathfrak{a} \subset \mathfrak{p}$, let $\exp(\mathfrak{a}) < G$ be the corresponding connected Lie group and let $A = \overline{\exp(\mathfrak{a})}$. We want to show that $\exp(\mathfrak{a})$ is closed and hence a torus.

Note that since $\mathfrak{a} \subset \mathfrak{p}$, then for all $X \in \mathfrak{a}$ $\Theta(X) = -X$. Moreover, since $\Theta = \mathrm{d}_e\sigma$ we have for all $X \in \mathfrak{a}$,

$$\sigma(\exp(X)) = \exp\big(\Theta(X)\big) = \exp(-X) = \exp(X)^{-1} ..$$

By continuity,

$$\sigma(a) = a^{-1} \qquad \forall a \in A$$

and therefore $\Theta(X) = -X$ for all $X \in \mathrm{Lie}(A)$. Thus $\mathrm{Lie}(A) \subset \mathfrak{p}$ and, by maximality of $\mathfrak{a}$ $\mathrm{Lie}(A) = \mathfrak{a}$. Thus $A = \exp(\mathfrak{a})$ is closed. Since $\exp(\mathfrak{a})$ is compact and any connected Abelian Lie group is of the form $\mathbb{T}^k \times \mathbb{R}^l$, $A = \exp(\mathfrak{a})$ must actually be a torus, since it is compact. There exists a dense flow, that is, $X \in \mathfrak{a}$ such that $\{\exp(tX) : t \in \mathbb{R}\}$ is dense in A. If $X \in \mathfrak{a} \subset \mathfrak{p}$, then $\mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$.

To show the reverse inclusion, let $Y \in \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ and note that $\{\exp(sY) : s \in \mathbb{R}\}$ commutes with $\{\exp(tX) : t \in \mathbb{R}\}$ and hence with $A = \exp(\mathfrak{a})$ by density. Thus $[Y, \mathfrak{a}] = 0$. Since $\mathfrak{a}$ is Abelian, $\mathfrak{a} + \mathbb{R}Y$ is Abelian too and $\mathfrak{a} + \mathbb{R}Y \subset \mathfrak{p}$. Since $\mathfrak{a}$ is maximal Abelian, we have that $\mathfrak{a} + \mathbb{R}Y \subset \mathfrak{a}$ which gives $Y \in \mathfrak{a}$.

Let now M be of non-compact type and $(\mathfrak{g}, \Theta)$ be the associated orthogonal symmetric Lie algebra. Take then M^* its compact dual with $(\mathfrak{g}^*, \Theta^*)$ the associated orthogonal symmetric Lie algebra. We note that $\mathfrak{a} \subset \mathfrak{p}$ is Abelian if and only if $i\mathfrak{a} \subset i\mathfrak{p}$ is Abelian. Therefore, by the conclusion for the compact case, there exists $iX \in i\mathfrak{p}$ such that $i\mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(iX) \cap i\mathfrak{p}$. Thus we conclude that

$$\mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(iX) \cap \mathfrak{p}. \qquad \blacksquare$$

Theorem III.8

Let $M = G/K$ be a Riemannian symmetric space of compact or non-compact type. If $\mathfrak{a}, \mathfrak{a}'$ are maximal Abelian subspaces of $\mathfrak{p}$, there exists $k \in K$ such that $\mathfrak{a}' = \mathrm{Ad}_G(k)\mathfrak{a}$.

Definition: Rank

Let M be a Riemannian symmetric space of compact or non-compact type. The **rank** $\mathrm{rk}(M)$ is the dimension of a maximal flat in M.

Proof of Theorem III.8. Let $X \in \mathfrak{a}$ and $X' \in \mathfrak{a}'$ be two regular elements. Consider the function

$$\begin{aligned} f\colon K &\longrightarrow \quad \mathbb{R} \\ k &\mapsto B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k)X, X') \end{aligned}$$

As K is compact and f is smooth, f has a critical point $k_0 \in K$. Thus for all $Z \in \mathfrak{k}$

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} & f(k_0 \exp(tZ)) = \\ = \left.\frac{d}{dt}\right|_{t=0} & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0 \exp(tZ))X, X') \\ = \left.\frac{d}{dt}\right|_{t=0} & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)\mathrm{Ad}_{\mathrm{G}}(\exp(tZ))X, X') \\ &= B_{\mathfrak{g}}\left(\mathrm{Ad}_{\mathrm{G}}(k_0) \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}_{\mathrm{G}}(\exp(tZ)X), X'\right) \\ &= B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)(\mathrm{ad}_{\mathfrak{g}} Z)X, X') \\ &= B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)[Z, X], X') \\ &= B_{\mathfrak{g}}([\mathrm{Ad}_{\mathrm{G}}(k_0)Z, \mathrm{Ad}_{\mathrm{G}}(k_0)X], X') \\ &= -B_{\mathfrak{g}}([\mathrm{Ad}_{\mathrm{G}}(k_0)X, \mathrm{Ad}_{\mathrm{G}}(k_o)Z], X') \\ &= B_{\mathfrak{g}}(\underbrace{\mathrm{Ad}_{\mathrm{G}}(k_0)Z}_{\in \mathfrak{k}}, \underbrace{[\mathrm{Ad}_{\mathrm{G}}(k_0)X, X']}_{\in \mathfrak{k}})\,. \end{aligned}$$

Since $\mathfrak{p}$ is $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant, then $\mathrm{Ad}_{\mathrm{G}}(k_0)X \in \mathfrak{p}$ and hence $[\mathrm{Ad}_{\mathrm{G}}(k_0)X, X'] \in \mathfrak{k}$. Since Z is arbitrary, it follows from the non-degeneracy of the Killing form that $[\mathrm{Ad}_{\mathrm{G}}(k_0)X, X'] = 0$, that is $\mathrm{Ad}_{\mathrm{G}}(k_0)X \in \mathrm{Centr}_{\mathfrak{g}}(X')$. Since X' is regular, this implies that $\mathrm{Ad}_{\mathrm{G}}(k_0)X \in \mathrm{Centr}_{\mathfrak{g}}(X') \cap \mathfrak{p} = \mathfrak{a}'$. But $\mathfrak{a}'$ is Abelian, hence every element in $\mathfrak{a}'$ commutes with $\mathrm{Ad}_{\mathrm{G}}(k_0)X$ and hence

$$\begin{aligned} \mathfrak{a}' &\subset \mathrm{Centr}_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)X) \cap \mathfrak{p} \\ &= \mathrm{Ad}_{\mathrm{G}}(k_0)(\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}) \\ &= \mathrm{Ad}_{\mathrm{G}}(k_0)(\mathfrak{a}). \end{aligned}$$

By maximality it follows that $\mathfrak{a}' = \mathrm{Ad}_{\mathrm{G}}(k_0)(\mathfrak{a})$. ■

Corollary III.9

Let M be a Riemannian symmetric space of compact or non-compact type. Every geodesic is contained in at least one maximal flat.
It is contained in exactly one flat if and only if the $X \in \mathfrak{p}$ that defines it is regular.

Proof. Let $\gamma(t) := \exp(tX)\cdot o$. Thus $\gamma\subset\exp(\mathfrak{a})\cdot o$ for every Abelian $\mathfrak{a}$ with $X\in\mathfrak{a}$.

($\Leftarrow$) Let $X\in\mathfrak{p}$ be regular and let $\mathfrak{a},\mathfrak{a}'\subset\mathfrak{p}$ be maximal Abelian subspaces such that $\gamma\subset\exp\mathfrak{a}\cdot o$ and $\gamma\subset\exp\mathfrak{a}'\cdot o$. Since $X\in\mathfrak{a}'$ and $\mathfrak{a}'$ Abelian then all elements in $\mathfrak{a}'$ commute with X and hence $\mathfrak{a}'\subset\mathrm{Centr}_{\mathfrak{g}}(X)\cap\mathfrak{p}=\mathfrak{a}$. But from Theorem III.8 we know that $\dim\mathfrak{a}=\dim\mathfrak{a}'$, so that from $\mathfrak{a}'\subseteq\mathfrak{a}$ we deduce that $\mathfrak{a}=\mathfrak{a}'$.

($\Rightarrow$) Conversely, let us suppose that γ is contained in exactly one flat, $\gamma\subset\exp\mathfrak{a}\cdot o$, where $\mathfrak{a}\subset\mathfrak{p}$ is maximal Abelian. Suppose that X is not regular, that is $\mathrm{Centr}_{\mathfrak{g}}(X)\cap\mathfrak{p}$ is not maximal Abelian, so that $\mathfrak{a}\subsetneq\mathrm{Centr}_{\mathfrak{g}}(X)\cap\mathfrak{p}$. Let $X'\in\mathrm{Centr}_{\mathfrak{g}}(X)\cap\mathfrak{p}$ and $X'\notin\mathfrak{a}$. Since $[X,X']=0$, the $\mathrm{Span}\{X,X'\}$ is Abelian and let $\mathfrak{a}'$ be a maximal Abelian that contains $\mathrm{Span}\{X,X'\}$. Since $X\in\mathfrak{a}'$, then $\gamma\subset\exp(\mathfrak{a}')\cdot o$ which is a contradiction because $X'\notin\mathfrak{a}$ and $X'\in\mathfrak{a}$, so that $\mathfrak{a}\neq\mathfrak{a}'$. ■

Example. We have seen that if $M=\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, then $\mathfrak{p}=\{X\in\mathfrak{sl}(n,\mathbb{R}): X=X^t\}$. We claim that $H\in\mathfrak{a}$ is regular if and only if $t_i\neq t_j$ if $i\neq$. Note that saying that $X\in\mathrm{Centr}_{\mathfrak{g}}(H)\cap\mathfrak{p}$ means that for all $1\leq i,j\leq n$

$$0=[H,X]_{ij}=(t_j-t_i)X_{ij}\,.$$

If $t_j\neq t_i$ for $i\neq j$, then $X_{ij}=0$, so that $X\in\mathfrak{a}$ and hence $\mathrm{Centr}_{\mathfrak{g}}(H)\cap\mathfrak{p}\subseteq\mathfrak{a}$. If, on the other hand, $0=[H,X]$ for some other H such that $t_i=t_j$ for some $i\neq j$, then X_{ij} could be non-zero and $\mathrm{Centr}_{\mathfrak{g}}(H)\cap\mathfrak{p}$ would not be Abelian anymore. Thus is

$$\mathfrak{a}:=\left\{\mathrm{diag}(t_1,\dots,t_n):\ t_j\in\mathbb{R},\sum_{j=1}^n t_j=0\right\}$$

is a maximal Abelian subspace of $\mathfrak{p}$.

We have seen that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is the set $\mathrm{Pos}_1(n)$ of positive matrices with determinant 1, with the action by conjugation $g\cdot p=g^tpg$ for $p\in\mathrm{Pos}_1(n)$, $g\in\mathrm{SL}(n,\mathbb{R})$ and with base point $Id_n\in\mathrm{Pos}_1(n)$. Hence a maximal flat is

$$\begin{aligned}F=\exp\mathfrak{a}\cdot Id_n&=\left\{\mathrm{diag}(e^{2t_1},\dots,e^{2t_n}):\ t_j\in\mathbb{R},\sum_{j=1}^n t_j=0\right\}\\&=\left\{\mathrm{diag}(\lambda_1,\dots,\lambda_n):\ \lambda_j>0,\prod_{j=1}^n\lambda_j=1\right\}.\end{aligned}$$

Now we show that if $X\in\mathfrak{a}$ has non-distinct eigenvalues, it is contained in a 1-parameter family of flats. We will show it in the case $n=3$. Consider

$$X\ =\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\in\mathfrak{a}$$

and let Y be a vector orthogonal to X (e.g. $Y = \mathrm{diag}(1,-1,0)$). Then $\mathfrak{a} = \mathrm{Span}\{X, Y\}$ is a maximal Abelian subalgebra containing $\gamma = \exp(tX) \subset \exp(\mathfrak{a})$. It is easy to see that if

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K,$$

then $\mathrm{Ad}_{\mathrm{G}}(k_\theta)X = X$, but $\mathrm{Ad}_{\mathrm{G}}(k_\theta)Y \neq Y$. Thus γ belongs to the one-parameter family of flats $\mathrm{Ad}_{\mathrm{G}}(k_\theta)\mathrm{Span}\{X, \mathrm{Ad}_{\mathrm{G}}(k_\theta)Y\} = \mathrm{Span}\{X, \mathrm{Ad}_{\mathrm{G}}(k_\theta)Y\}$.

Example. Let $M = \mathrm{SO}(p,q)/SO(p) \times \mathrm{SO}(q)$, for $p \leq q$. A maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} : A = (a_{ij}) \in M_{p\times q}(\mathbb{R}),\ a_{ij} = 0 \text{ if } i \neq j \right\},$$

so $\mathrm{rk}(\mathrm{SO}(p,q)/SO(p) \times \mathrm{SO}(q)) = \min\{p,q\}$.

Example. If $M = \mathrm{Sp}(2q,\mathbb{R})/(\mathrm{SO}(2q) \cap \mathrm{Sp}(2q,\mathbb{R}))$, then a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1, \ldots, t_q),\ t_j \in \mathbb{R} \right\}.$$

and hence $\mathrm{rk}(M) = q$.

III.3 Roots and Root Spaces

Let $(\mathfrak{g}, \Theta)$ be an orthogonal symmetric Lie algebra of non-compact type. On $\mathfrak{g} \times \mathfrak{g}$ we can define the following positive definite bilinear form

$$\langle X, Y \rangle \coloneqq -B_{\mathfrak{g}}(X, \Theta(Y)) . \tag{III.3}$$

Notice that the restriction of this form to $\mathfrak{p}$ coincides with the Killing form.

Lemma III.10:

For every $X \in \mathfrak{p}$ the operator $\mathrm{ad}_{\mathfrak{g}} X$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$.

Proof. We need to show that if $X \in \mathfrak{p}$, $Y, Z \in \mathfrak{g}$, then

$$\langle (\mathrm{ad}_{\mathfrak{g}} X)Y, Z \rangle = \langle Y, (\mathrm{ad}_{\mathfrak{g}} X)Z \rangle .$$

This is a simple verification. In fact, from $\Theta(X) = -X$ we get

$$\begin{aligned}\langle(\operatorname{ad}_{\mathfrak{g}} X)Y, Z\rangle &= -B_{\mathfrak{g}}((\operatorname{ad}_{\mathfrak{g}} X)(Y), \Theta(Z)) \\ &= B_{\mathfrak{g}}(Y, (\operatorname{ad}_{\mathfrak{g}} X)\Theta(Z)) \\ &= B_{\mathfrak{g}}(Y, [X, \Theta(Z)]) \\ &= B_{\mathfrak{g}}(Y, [-\Theta(X), \Theta(Z)]) \\ &= -B_{\mathfrak{g}}(Y, \Theta((\operatorname{ad}_{\mathfrak{g}} X)Z)) \\ &= \langle Y, (\operatorname{ad} X)Z\rangle .\end{aligned}$$

■

It follows from the above lemma that if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subspace, then $\{\operatorname{ad}_{\mathfrak{g}} X : X \in \mathfrak{a}\}$ is a commuting family of self-adjoint operators which are simultaneously diagonalisable, and we can consider the following

Definition: Root and Root space

A linear map $\alpha\colon \mathfrak{a} \to \mathbb{R}$ is called a **root** of the pair $(\mathfrak{g}, \mathfrak{a})$ if

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : (\operatorname{ad}_{\mathfrak{g}} H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \neq \{0\}.$$

In this case the subspace $\mathfrak{g}_\alpha$ is called a **root space**.

We have the following rather immediate properties:

Lemma III.11

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(2) $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, *and, in fact,* $\Theta\colon \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ *is an isomorphism.*

Proof. (1) Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then for any $H \in \mathfrak{a}$ we have $[H, X] = \alpha(H)X$ and $[H, Y] = \beta(H)Y$, so that, by the Jacobi identity,

$$\begin{aligned}[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] &= [\alpha(H)X, Y] + [X, \beta(H)Y] \\ = \alpha(H)[X, Y] + \beta(H)[X, Y] &= (\alpha + \beta)(H)[X, Y] .\end{aligned}$$

(2) Let $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{a} \subset \mathfrak{p}$ arbitrary. Then $[H, X] = \alpha(H)X$ and $\Theta(H) = -H$, so that

$$[H, \Theta(X)] = -[\Theta(H), \Theta(X)] = -\Theta[H, X] = -\Theta(\alpha(H)X) = -\alpha(H)\Theta(X) .$$

■

If $\alpha \equiv 0$, then $\mathfrak{a} \subset \mathfrak{g}_0 = \operatorname{Centr}_{\mathfrak{g}}(\mathfrak{a})$. If

$$\Sigma := \{\alpha : \alpha \text{ is a non-trivial root of } (\mathfrak{g}, \mathfrak{a})\}$$

denotes the set of non-trivial roots of $(\mathfrak{g}, \mathfrak{a})$, then we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha \,,$$

because $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a commuting family of diagonalisable endomorphisms. Observe that this decomposition is orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$. Moreover it follows from the finite dimensionality of $\mathfrak{g}$ that the set Σ is finite.

Before showing that the roots of the pair $(\mathfrak{g}, \mathfrak{a})$ satisfy the properties of an **abstract root system** (see Theorem III.15 and § **??**), we want to show how regular elements can be found using the roots of $(\mathfrak{g}, \mathfrak{a})$.

Notation. From now on, we will always assume that $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subalgebra. Then we observe

Lemma III.12

If $\mathfrak{a} \subset \mathfrak{p}$ is maximal Abelian, then

$$\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}.$$

Proof. We obviously have

$$\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p} = \{X \in \mathfrak{p} : [X, H] = 0 \ \text{ for all } H \in \mathfrak{a}\}.$$

On the other hand, if $X \in \mathfrak{g}_0 \cap \mathfrak{p}$, then $\mathfrak{a} + \mathbb{R}X$ is an Abelian subspace containing $\mathfrak{a}$, which, by maximality of $\mathfrak{a}$, implies that $X \in \mathfrak{a}$. ■

Lemma III.13:

A vector $H \in \mathfrak{a} \setminus \{0\}$ is regular if and only if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$.

Proof. $(\Rightarrow)$ As $H \in \mathfrak{a} \setminus \{0\}$ is regular, $\mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ is maximal Abelian. Since $\mathfrak{a}$ is maximal Abelian and $\mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ we necessarily have $\mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$.

Assume by contradiction that there exists $\alpha \in \Sigma$ such that $\alpha(H) = 0$. Let $X \in \mathfrak{g}_\alpha \setminus \{0\}$, $X_{\mathfrak{k}} = \frac{1}{2}(X + \Theta(X))$ and $X_{\mathfrak{p}} = \frac{1}{2}(X - \Theta(X))$. Then $X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$ and we get for any $A \in \mathfrak{a}$

$$\mathrm{ad}_{\mathfrak{g}}(A)(X_{\mathfrak{k}} + X_{\mathfrak{p}}) = \alpha(A)(X_{\mathfrak{k}} + X_{\mathfrak{p}}).$$

From the relations $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ it follows in particular that for all $A \in \mathfrak{a}$

$$\mathrm{ad}_{\mathfrak{g}}(A)(X_{\mathfrak{p}}) = \alpha(A) X_{\mathfrak{k}} \,. \tag{III.4}$$

But if $\alpha(H) = 0$, then $\operatorname{ad}_{\mathfrak{g}}(H)(X_{\mathfrak{p}}) = 0$, hence $X_{\mathfrak{p}} \in \operatorname{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$. From this and again from (III.4) we get

$$0 = \operatorname{ad}_{\mathfrak{g}}(A)(X_{\mathfrak{p}}) = \alpha(A) X_{\mathfrak{k}}$$

for all $A \in \mathfrak{a}$. But as $\alpha \not\equiv 0$ on $\mathfrak{g}_\alpha$ this implies $X_{\mathfrak{k}} = 0$ and therefore $X = X_{\mathfrak{p}} \in \mathfrak{a}$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{a}$ in contradiction to the fact that $\mathfrak{a} \subset \mathfrak{g}_0$.

($\Longleftarrow$) Assume now that $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, but that $H \in \mathfrak{a} \setminus \{0\}$ is not regular. Then $\operatorname{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ is not maximal Abelian, that is $\mathfrak{a} \subsetneq \operatorname{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$. So there exists $Y \in \operatorname{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ with $Y \notin \mathfrak{a}$. Let Y_α denote the projection of Y on the root subspace $\mathfrak{g}_\alpha$, so that

$$Y = Y_0 \oplus \sum_{\alpha \in \Sigma} Y_\alpha,$$

where $Y_0 \in \operatorname{Centr}_{\mathfrak{g}}(\mathfrak{a}) \subset \operatorname{Centr}_{\mathfrak{g}}(H)$. Then

$$0 = [H, Y] = [H, \sum_{\alpha \in \Sigma} Y_\alpha] = \sum_{\alpha \in \Sigma} [H, Y_\alpha] = \sum_{\alpha \in \Sigma} \alpha(H) Y_\alpha \,.$$

Since $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$ this implies $Y_\alpha = 0$ for all $\alpha \in \Sigma$. We conclude $Y = Y_0 \in \operatorname{Centr}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{p} = \mathfrak{a}$ which is a contradiction. ■

Corollary III.14

Let $\mathfrak{a}_{\mathrm{reg}}$ denote the set of regular elements in $\mathfrak{a}$. Then

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha \,.$$

Definition: Weyl Chamber

Let $\mathfrak{a}$ be a maximal Abelian subalgebra. A connected component of $\mathfrak{a}_{\mathrm{reg}}$ is called a **Weyl chamber** in $\mathfrak{a}$.

Remark.

- Note that a Weyl chamber is an open cone in Euclidean space, as it is the complement of a collection of hyperplanes

$$\{X \in \mathfrak{g} : \alpha(X) = 0\} \,.$$

- A Weyl chamber can also be described as the equivalence classes in $\mathfrak{a}$ of the equivalence relation

$$H_1 \sim H_2 \qquad :\Longleftrightarrow \qquad \alpha(H_1)\alpha(H_2) > 0 \;\text{ for all }\; \alpha \in \Sigma \,.$$

Example (Continuation of Example III.2)**.** Let $G/K = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and denote by E_{ij} the matrix whose (i,j)-th matrix coefficient is 1 and all other matrix coefficients are 0. If $H_j := E_{jj} - E_{j+1,j+1}$, then $(H_1,\dots,H_{n-1}, E_{ij} : 1 \leq i \neq j \leq n)$ is a basis for $\mathfrak{g}$. Moreover, if $H = \mathrm{diag}(t_1,\dots,t_n) = \sum_{j=1}^n t_j E_{jj} \in \mathfrak{a}$, then it is easy to check that

$$\mathrm{ad}_{\mathfrak{g}}(H)(E_{ij}) = [H, E_{ij}] = (t_i - t_j)E_{ij}\,,$$

and

$$\mathrm{ad}_{\mathfrak{g}}(H)(H_j) = 0$$

for all i,j. Thus there are $n(n-1)$ non-zero roots $\{\alpha_{ij}\}_{i\neq j}$, given by

$$\alpha_{ij}(A) = A_{ii} - A_{jj}\,,$$

and $n(n-1)$ one-dimensional root spaces $\mathfrak{g}_{ij} := \mathfrak{g}_{\alpha_{ij}}$ spanned by E_{ij} for $i \neq j$. The space $\mathfrak{a}$ is spanned by $(H_1,\dots,H_{n-1})$ and $\mathfrak{g}_0 = \mathfrak{a}$. In particular we can write

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{a} \oplus \sum_{i\neq j} \mathbb{R}E_{ij}\,.$$

We want to show in this example that there is a one-to-one correspondence between the Weyl chambers of $\mathfrak{a}$ and the elements of the permutation group S_n in n letters. In fact, let $A = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ and $B = \mathrm{diag}(\mu_1,\dots,\mu_n)$ be regular elements in $\mathfrak{a}$. Since A is regular, the λ_j are distinct and there exists a unique permutation $\sigma \in S_n$ such that

$$\lambda_{\sigma(1)} > \cdots > \lambda_{\sigma(n)}\,.$$

Similarly, since B is regular, there exists a unique permutation $\tau \in S_n$ such that

$$\mu_{\tau(1)} > \cdots > \mu_{\tau(n)}\,.$$

The condition that A, B determine the same Weyl chamber is exactly that they are equivalent, that is

$$(\lambda_i - \lambda_k)(\mu_i - \mu_k) > 0$$

for all $i \neq k$. It is not difficult to show that this holds if and only if $\sigma = \tau$, so that a Weyl chamber in $\mathfrak{a}$ is given by

$$\mathfrak{a}^+ := \{\mathrm{diag}(t_1,\dots,t_n) \in \mathfrak{a} : \sum_{i=1}^n t_i = 0,\, t_1 > t_2 > \cdots > t_n\}\,.$$

Now let

$$\begin{aligned} \mathfrak{a}^* &\longrightarrow \mathfrak{a} \\ \alpha &\mapsto H_\alpha \end{aligned}$$

be the isomorphism defined by

$$\alpha(H) = B_{\mathfrak{g}}(H, H_\alpha) \qquad \text{for all} \quad H \in \mathfrak{a}.$$

The central result is the following

Theorem III.15

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subspace, $\Sigma \subset \mathfrak{a}^* \setminus \{0\}$ the set of non-trivial roots of $\mathfrak{a}$ in $\mathfrak{g}$. Then Σ is a ***root system****, that is*

(1) Σ spans $\mathfrak{a}^*$. Let $\alpha, \beta \in \Sigma$. Then

(2)

$$\beta - \frac{2B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\alpha \in \Sigma \text{ and}$$

(3)

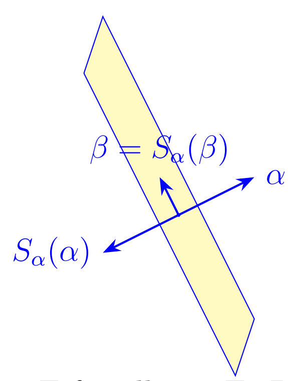
$$2\,\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \in \mathbb{Z}.$$

Remark. We note that for every $\alpha \in \mathfrak{a}^* \setminus \{0\}$ the map

$$\begin{aligned} S_\alpha \colon \mathfrak{a}^* &\longrightarrow \mathfrak{a}^* \\ \beta &\mapsto \beta - \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha \end{aligned}$$

is the orthogonal reflection fixing the hyperplane orthogonal to α. In particular we have

- $S_\alpha(\alpha) = -\alpha$ and
- $S_\alpha(\beta) = \beta$ if $\beta \perp \alpha$.



and property (2) above states that $S_\alpha(\Sigma) \subset \Sigma$ for all $\alpha \in \Sigma$. Property (3) above is more mysterious and will be a consequence of the classification of finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$.

We start the proof of Theorem III.15 with the following lemma:

Lemma III.16

Let $(\mathfrak{g},\Theta)$ be an orthogonal symmetric Lie algebra of non-compact type and $\mathfrak{a}$ a maximal Abelian subspace of $\mathfrak{p}$. For $\alpha\in\Sigma$ and $X\in\mathfrak{g}_\alpha$ we denote x_α the unique positive multiple of X such that

$$\langle x_\alpha, x_\alpha\rangle = \frac{2}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}.$$

Let

$$y_\alpha = -\Theta(x_\alpha) \qquad \text{and} \qquad h_\alpha = \frac{2H_\alpha}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}.$$

Then

$$\begin{aligned}[x_\alpha,y_\alpha] &= h_\alpha\\ [h_\alpha,x_\alpha] &= 2x_\alpha\\ [h_\alpha,y_\alpha] &= -2y_\alpha.\end{aligned}$$

Example. Let us now look at a special case of Example III.2, namely $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$ and $\Theta\in\mathrm{Aut}(\mathfrak{g})$ defined by

$$\Theta(X) = -X^t \qquad \text{for} \quad X\in\mathfrak{sl}(2,\mathbb{R}).$$

Recall that

$$\mathfrak{a} = \left\{\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix} : \lambda\in\mathbb{R}\right\}$$

is a maximal Abelian subspace of $\mathfrak{p}$. Since for all λ, x, $y\in\mathbb{R}$

$$\begin{aligned}\left[\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix},\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\right] &= 2\lambda\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\\ \left[\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix},\begin{pmatrix}0 & 0\\ y & 0\end{pmatrix}\right] &= -2\lambda\begin{pmatrix}0 & 0\\ y & 0\end{pmatrix}\end{aligned}$$

defining $\alpha\in\mathfrak{a}^*$ by

$$\alpha\left(\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}\right) = 2\lambda$$

we get $\mathfrak{g}_0 = \mathfrak{a}$ and

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\},$$
$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} : y \in \mathbb{R} \right\}.$$

To find the H_α that realizes the isomorphism $\mathfrak{a}^* \to \mathfrak{a}$, $\alpha \mapsto H_\alpha$, let $H = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in \mathfrak{a}$ be a generic element and let us find $\mu \in \mathbb{R}$ such that $H_\alpha = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$ realizes the isomorphism. Then a computation shows that

$$2\lambda = \alpha(H) = B_\mathfrak{g}(H, H_\alpha) = 8\lambda\mu,$$

so that $\mu = \frac{1}{4}$ and hence

$$H_\alpha = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, \quad B_\mathfrak{g}(H_\alpha, H_\alpha) = 8\mu^2 = \frac{1}{2}, \text{ and } h_\alpha = \frac{2H_\alpha}{B_\mathfrak{g}(H_\alpha, H_\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we look for x_α. Let $X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\alpha$.A computation gives

$$\langle \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \rangle = 4x^2$$

and setting this to be equal to

$$\frac{2}{B_\mathfrak{g}(H_\alpha, H_\alpha)}.$$

gives us that $x_\alpha = \frac{1}{x} X$. Thus

$$x_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We found in fact the standard basis for $\mathfrak{sl}(2, \mathbb{R})$, namely

$$e_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The following corollary is the generalization of the above example to the case of a semisimple Lie algebra.

Corollary III.17

Given $\alpha \in \Sigma$ *and* $X \in \mathfrak{g}_\alpha \backslash \{0\}$ *we define* x_α, y_α *and* h_α *as in Lemma III.16. Then the linear map* $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ *defined on a basis via*

$$\begin{aligned} e_+ &\mapsto x_\alpha \\ e_- &\mapsto y_\alpha \\ h &\mapsto h_\alpha \end{aligned}$$

is an injective Lie algebra homomorphism with image

$$\mathfrak{sl}(2, \mathbb{R})_X := \mathbb{R}x_\alpha + \mathbb{R}h_\alpha + \mathbb{R}y_\alpha \subset \mathfrak{g}.$$

Remark. Note that $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{a}$.

Proof of Lemma III.16. Since $h_\alpha \in \mathfrak{a}$ and $x_\alpha \in \mathfrak{g}_\alpha$ we have

$$[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha.$$

By definition of h_α,

$$\begin{aligned} \alpha(h_\alpha) &= \frac{2\alpha(H_\alpha)}{B_\mathfrak{g}(H_\alpha, H_\alpha)} \\ &= \frac{2B_\mathfrak{g}(H_\alpha, H_\alpha)}{B_\mathfrak{g}(H_\alpha, H_\alpha)} \\ &= 2. \end{aligned}$$

Similarly we compute $[h_\alpha, y_\alpha] = -2y_\alpha$.

To show that $[x_\alpha, y_\alpha] = h_\alpha$, note that

$$[x_\alpha, y_\alpha] = [x_\alpha, -\Theta(x_\alpha)] = -[x_\alpha, \Theta(x_\alpha)].$$

Since

$$\begin{aligned} h_\alpha &= \frac{2H_\alpha}{B_\mathfrak{g}(H_\alpha, H_\alpha)} \\ &= \langle x_\alpha, x_\alpha \rangle H_\alpha \end{aligned}$$

the claim follows from the following statement. ■

Lemma III.18

Let $\alpha \in \Sigma, X \in \mathfrak{g}_\alpha$. *Then*

$$[X, \Theta(X)] = -\langle X, X \rangle H_\alpha.$$

Proof. By the definition (III.3) of the scalar product on $\mathfrak{g}$ it is enough to prove that

$$[X, \Theta(X)] = B_{\mathfrak{g}}(X, \Theta(X))H_\alpha.$$

We first claim that $[X, \Theta(X)] \in \mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$: Indeed, $X \in \mathfrak{g}_\alpha$ implies $\Theta(X) \in \mathfrak{g}_{-\alpha}$, hence $[X, \Theta(X)] \in \mathfrak{g}_{\alpha+(-\alpha)} = \mathfrak{g}_0$. On the other hand, we have

$$\Theta([X, \Theta(X)]) = [\Theta(X), X] = -[X, \Theta(X)],$$

hence also $[X, \Theta(X)] \in \mathfrak{p}$.

Therefore

$$[X, \Theta(X)] - B_{\mathfrak{g}}(X, \Theta(X))H_\alpha \in \mathfrak{a},$$

and it remains to prove that this expression is in fact zero. Our strategy of proof is to show that it is orthogonal to every element in $\mathfrak{a}$:

So let $H \in \mathfrak{a} \subset \mathfrak{p}$ be arbitrary. Then using the equation

$$\alpha(H) = B_{\mathfrak{g}}(H, H_\alpha) = B_{\mathfrak{g}}(H_\alpha, H) = \langle H_\alpha, H \rangle$$

we get

$$\begin{aligned}
\langle [X, \Theta(X)], H \rangle &= -B_{\mathfrak{g}}([X, \Theta(X)], \Theta(H)) \\
&= B_{\mathfrak{g}}([X, \Theta(X)], H) \\
&= -B_{\mathfrak{g}}(\Theta(X), [X, H]) \\
&= B_{\mathfrak{g}}(\Theta(X), [H, X]) \\
&= B_{\mathfrak{g}}(\Theta(X), \alpha(H)X) \\
&= \alpha(H) B_{\mathfrak{g}}(\Theta(X), X) \\
&= \langle H_\alpha, H \rangle B_{\mathfrak{g}}(\Theta(X), X) \\
&= \langle B_{\mathfrak{g}}(\Theta(X), X) H_\alpha, H \rangle
\end{aligned}$$

and it follows that

$$\langle [X, \Theta(X)] - B_{\mathfrak{g}}(\Theta(X), X)H_\alpha, H \rangle = 0 \qquad \text{for all} \quad H \in \mathfrak{a}. \qquad \blacksquare$$

Remark. $\alpha(h_\alpha) = 2, \alpha(H) = B_{\mathfrak{g}}(H, H_\alpha)$ will be used over and over in the upcoming computations.

According to Corollary III.17 we obtain for any $X \in \mathfrak{g}_\alpha \setminus \{0\}$ a representation of $\mathfrak{sl}(2, \mathbb{R})$ on $\mathfrak{g}$ via

$$\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})_X \overset{\mathrm{ad}_{\mathfrak{g}}}{\to} \mathfrak{gl}(\mathfrak{g}).$$

It is therefore essential to understand the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. We summarise the relevant information in the following

Theorem III.19: [Ebe96, Lemma 1, p. 151]

(1) Every finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ is a direct sum of irreducible representations.

(2) Every finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ is classified, up to isomorphisms, by its dimension. If $\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V)$ is an irreducible representation, then $\rho(h)$ is diagonalizable with simple eigenvalues

$$\{(\dim V - 1) - 2n : n = 0, \dots, \dim V - 1, \}$$

that is

$$\{-(\dim V - 1), -(\dim V - 3), \dots, \dim V - 3, \dim V - 1\}.$$

Examples. (1) If $\dim V = 1$, we have only the trivial representation $\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathbb{R}) = \mathbb{R}$ where $\rho(X) = 0$. In particular $\rho(h) = 0$ and 0 is the only eigenvalue.

(2) If $\dim V = 2$ we have the identity representation $\rho\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathbb{R}^2)$, $\rho(X) = X$. Since

$$\rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the eigenvalues are $-1, +1$ with eigenspaces $\mathbb{R}e_1$, $\mathbb{R}e_2 \subset \mathbb{R}^2$.

(3) The adjoint representation

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{sl}(2,\mathbb{R}) &\to \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R})) \\ X &\mapsto \mathrm{ad}(X) \end{aligned}$$

is the irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ in dimension 3. The eigenvalues are -2, 0, 2 with eigenspaces $\mathbb{R}e_-$, $\mathbb{R}e_+$, $\mathbb{R}h$.

(4) The general irreducible representation of dimension $n+1$ of $\mathfrak{sl}(2,\mathbb{R})$ can be described as follows. Let V_n be the $(n+1)$-dimensional vector space of homogeneous polynomials in X, Y of degree n, that is

$$V_n \coloneqq \Big\{\sum_{k=0}^{n} a_k X^k Y^{n-k} : a_k \in \mathbb{R}\Big\}.$$

Then the Lie group $\mathrm{SL}(2,\mathbb{R})$ acts on V_n by linear substitution via

$$\begin{aligned} \rho_n\colon\ \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{GL}(V_n) \\ g &\mapsto \rho(g) \end{aligned}$$

where $\big(\rho_n(g)P\big)(X,Y) := P\big((X,Y)g\big)$ for $P \in V_n$. Differentiating at the identity thus yields a map $\mathrm{d}_{Id}\rho_n \colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V_n)$.

Explicitly, writing $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$ we get

$$\begin{aligned}(\mathrm{d}_{Id}\rho_n)(A)P(X,Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_n\big(\exp(tA)\big)P(X,Y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P((X,Y)\exp(tA)) \\ &= \mathrm{d}_{(X,Y)}P\big((X,Y)A\big) \\ &= \frac{\partial P}{\partial X}(aX+cY) + \frac{\partial P}{\partial Y}(bX-aY)\end{aligned}$$

Taking $A = h$ and $P(X,Y) = X^k, Y^{n-k}$, this simplifies to

$$(\mathrm{d}_{Id}\rho_n)(h)X^kY^{n-k} = (2k-n)X^kY^{n-k}$$

so that the eigenvalues are $\{2k-n : k = 0,\dots,n\} = \{-n,-n+2,\dots,n-2,n\}$ with eigenspaces $\mathbb{R}Y^n$, $\mathbb{R}XY^{n-1},\dots,\mathbb{R}X^{n-1}Y$, $\mathbb{R}X^n$.

Proof of Theoren III.15. For $\alpha, \beta \in \Sigma$ we set

$$W_{\beta,\alpha} := \bigoplus \{\mathfrak{g}_{\beta+k\alpha} : k \in \mathbb{Z}\} \subset \mathfrak{g}.$$

For $X \in \mathfrak{g}_\alpha \setminus \{0\}$ we consider $\mathrm{ad}_\mathfrak{g}$ restricted to $\mathfrak{sl}(2,\mathbb{R})_X = \mathbb{R}x_\alpha + \mathbb{R}h_\alpha + \mathbb{R}y_\alpha$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ we get that $W_{\beta,\alpha}$ is invariant by $\mathrm{ad}_\mathfrak{g}\big|_{\mathfrak{sl}(2,\mathbb{R})_X}$. Moreover, if $Y \in \mathfrak{g}_{\beta+k\alpha}$, then

$$\begin{aligned}\mathrm{ad}_\mathfrak{g}(h_\alpha)(Y) &= [h_\alpha, Y] \\ &= (\beta+k\alpha)(h_\alpha)Y = \big(\beta(h_\alpha) + k\alpha(h_\alpha)\big)Y\end{aligned}$$

and thus the eigenvalues of $\mathrm{ad}_\mathfrak{g}(h_\alpha)$ on $W_{\beta,\alpha}$ are

$$\begin{aligned}&\{\beta(h_\alpha) + k\underbrace{\alpha(h_\alpha)}_{=2} : k \in \mathbb{Z} \text{ such that } \mathfrak{g}_{\beta+k\alpha} \neq \{0\}\} \\ &= \{\beta(h_\alpha) + 2k : k \in \mathbb{Z} \text{ such that } \beta + k\alpha \in \Sigma \cup \{0\}\}.\end{aligned}$$

Since $\mathfrak{g}$ is finite dimensional, there is of course a finite number of eigenvalues of $\mathrm{ad}(h_\alpha)$ on $W_{\beta,\alpha}$. We set then

$$\begin{aligned}r &:= \min\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\} \\ s &:= \max\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\}\end{aligned}$$

and let $n_X := n_X(\beta, \alpha)$ be the maximal dimension of an irreducible submodule of $W_{\beta,\alpha}$ that is $\mathfrak{sl}(2,\mathbb{R})_X$-invariant. By Theorem III.19 (2) we have

$$\begin{aligned} 1 - n_X &= \beta(h_\alpha) + 2r \\ n_X - 1 &= \beta(h_\alpha) + 2s \end{aligned}$$

and therefore $-\beta(h_\alpha) = r + s \in \mathbb{Z}$, which shows assertion (3) in Theorem III.15 since

$$\begin{aligned} \beta(h_\alpha) &= \beta\left(\frac{2H_\alpha}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\right) \\ &= 2\frac{\beta(H_\alpha)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \\ &= 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}. \end{aligned}$$

Again by the theory of irreducible finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ we note that $\mathfrak{g}_{\beta+k\alpha} \neq \{0\}$ if and only if $r \leq k \leq s$.

As $\beta \in \Sigma$ we know that $\mathfrak{g}_\beta \neq \{0\}$ and thus $r \leq 0 \leq s$. Since $-\beta(h_\alpha) = r + s$ we thus further have $r \leq -\beta(h_\alpha) \leq s$ and consequently $\mathfrak{g}_{\beta-\beta(h_\alpha)\alpha} \neq \{0\}$. This leaves two possibilities:

(i) If $\beta - \beta(h_\alpha)\alpha \neq 0$, then $\beta - \beta(h_\alpha) \in \Sigma$.

(ii) If $\beta - \beta(h_\alpha)\alpha = 0$, then $\beta = \beta(h_\alpha)\alpha$. In particular $\beta(h_\alpha) = \beta(h_\alpha)\alpha(h_\alpha) = 2\beta(h_\alpha)$, which implies $\beta(h_\alpha) = 0$ so that $\beta - \beta(h_\alpha)\alpha = \beta \in \Sigma$.

In any case we have $\beta - \beta(h_\alpha)\alpha \in \Sigma$, so assertion (2) in Theorem III.15 is proved. ■

In fact, in the course of the previous proof we have shown the following fact of independent interest:

Lemma III.20

Let α, $\beta \in \Sigma$ and assume that $\beta \notin \mathbb{Z}\alpha$. Define

$$\begin{aligned} r &:= \min\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\}, \\ s &:= \max\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\}. \end{aligned}$$

Then for all $k \in [r, s] \cap \mathbb{Z}$ we have $\beta + k\alpha \in \Sigma$.

We can hence give the following definition

Definition

Let $\alpha, \beta \in \Sigma$. An α**-string of** β is a subset of Σ of the form

$$\{\beta + k\alpha : r \leq k \leq s,\ r, s \in \mathbb{Z}\}.$$

The α-string of β is **maximal** if $\beta + (r-1)\alpha \notin \Sigma$ and $\beta + (s+1)\alpha \notin \Sigma$.

III.4 Abstract Root Systems

Remark. Root systems are very useful due to several aspects such as

- the classification of complex semisimple Lie algebras,
- the study of the finite dimensional representations of semisimple Lie groups and
- the study of geometric properties of Riemannian symmetric space .

Theorem III.15 introduces a structure called root system that leads to a finite reflection group, its Weyl group. The latter is an example of a much wider class of groups called Coxeter groups that acquired a prominent status in the theory of buildings and in geometric group theory. In this section we will establish certain fundamental properties of root systems and their Weyl group.

Let $\mathbb{E}$ be a euclidean space with scalar product $\langle \cdot, \cdot \rangle$, and $\gamma \in \mathbb{E} \setminus \{0\}$. Recall that

$$\sigma_\gamma(\alpha) := \alpha - \frac{2\langle \alpha, \gamma \rangle}{\langle \gamma, \gamma \rangle}\gamma$$

is the reflection on $\mathbb{E}$ with respect to the hyperplane $\gamma^\perp$. In particular we have $\sigma_\gamma(\gamma) = -\gamma$.

Definition: Root systems

A **root system of rank** $\dim \mathbb{E}$ is a subset $\Sigma \subset \mathbb{E} \setminus \{0\}$ such that

(1) Σ spans $\mathbb{E}$,

(2) $\sigma_\alpha(\Sigma) = \Sigma$ for all $\alpha \in \Sigma$,

(3) for all $\alpha, \beta \in \Sigma$ we have $2\dfrac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Example. The root spaces of $\mathfrak{a}$ in $\mathfrak{g}$.

Remark. The third condition implies that if $\beta = \lambda\alpha \in \Sigma$, then $\lambda \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$, because these are the only values for which $2\lambda \in \mathbb{Z}$ and $\frac{2}{\lambda} \in \mathbb{Z}$.

Definition: Reduced Root Systems

A root system is **reduced** if from $\beta = \lambda\alpha$ it follows that $\lambda = +1$ or $\lambda = -1$.

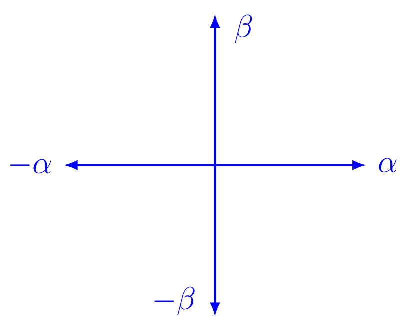
Remark. Given an arbitrary root system Σ, on obtains a reduced root system by setting

$$\Sigma' := \left\{ \alpha \in \Sigma : \frac{\alpha}{2} \neq \Sigma \right\}.$$

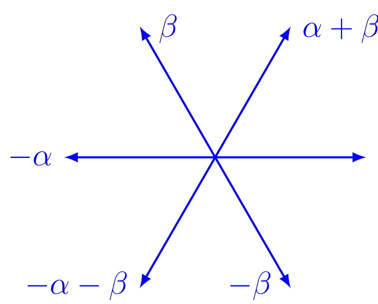
Since our focus will be on the group generated by the reflections $\{\sigma_\alpha : \alpha \in \Sigma\}$, we may restrict ourselves to the reduced root system Σ', because the reflections $\{\sigma_\alpha : \alpha \in \Sigma'\}$ generate the same group.

Example. Examples in rank 2 are

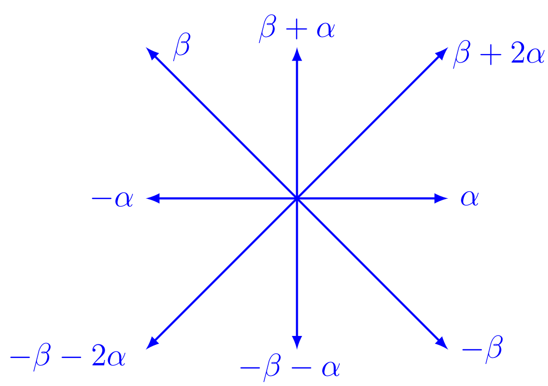
- $A_1 \times A_1$:



- A_2:



- B_2:



We want to understand the configuration of root systems. If we call $\varphi \coloneqq \measuredangle(\alpha, \beta)$, then we observe that

$$\begin{aligned} n(\beta, \alpha) &\coloneqq 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \\ &= 2\frac{\|\beta\|}{\|\alpha\|} \cos\varphi. \end{aligned}$$

But it follows from the third condition that

$$n(\beta, \alpha) = 2\frac{\|\beta\|}{\|\alpha\|} \cos\varphi \in \mathbb{Z}$$

and reversing the roles of α and β also that

$$n(\alpha, \beta) = 2\frac{\|\alpha\|}{\|\beta\|} \cos\varphi \in \mathbb{Z}.$$

It follows that $n(\alpha, \beta)n(\beta, \alpha) = 4\cos^2\varphi \in \mathbb{Z}$, which implies $\cos\varphi \in \{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}\}$ and hence only leaves the following possibilites for non-proportional roots:

φ	$n(\beta, \alpha)$	$n(\alpha, \beta)$	$\frac{\|\beta\|^2}{\|\alpha\|^2}$
$\frac{\pi}{2}$	0	0	undetermined
$\frac{\pi}{3}$	1	1	1
$\frac{2\pi}{3}$	-1	-1	1
$\frac{\pi}{4}$	2	1	2
$\frac{3\pi}{4}$	-2	-1	2
$\frac{\pi}{6}$	3	1	3
$\frac{5\pi}{6}$	-3	-1	3

Out of this table we get the following

Lemma III.21

If $\alpha, \beta \in \Sigma$ are not proportional, then

- *if $\langle \alpha, \beta \rangle > 0$, we have $\beta - \alpha \in \Sigma$, and*
- *if $\langle \alpha, \beta \rangle < 0$, we have $\beta + \alpha \in \Sigma$.*

Definition: Basis of Root System

A subset $\Delta \subset \Sigma$ is called a **basis** for Σ if

(1) Δ is a basis of $\mathbb{E}$,

(2) for any root $\alpha \in \Sigma$ all coordinates of α in the basis Δ are integers of the same sign.

Definition: Simple Roots

The elements of Δ are called simple roots.

Lemma III.22

If $\alpha, \beta \in \Delta$ are two distinct simple roots, then $\langle \alpha, \beta \rangle \leq 0$.

Proof. If $\langle \alpha, \beta \rangle > 0$, then $\beta - \alpha \in \Sigma$ by the previous Lemma. This contradicts property (2) of a basis. ■

Notation. For $\gamma \in \mathbb{E} \backslash \{0\}$ we write

$$P_\gamma = \{\beta \in \mathbb{E} : \langle \gamma, \beta \rangle = 0\}$$

Notice that P_γ is a hyperplane which separates $\mathbb{E}$ into two connected components.

Definition: Weyl Chamber and Regular Elements

- A **Weyl Chamber** is a connected component of $\mathbb{E} \backslash \bigcup_{\alpha \in \Sigma} P_\alpha$.
- An element $\gamma \in \mathbb{E}$ is **regular** if $\langle \alpha, \gamma \rangle \neq 0$ for all roots $\alpha \in \Sigma$, that is if γ belongs to some Weyl chamber.

Remark. This is compatible with the previous definitions.

THE FOLLOWING NEEDS TO BE CHECKED AND MAYBE PUT AT THE END OF THE SECTION...

Definition: Coxeter Graph

- A **Coxeter Graph** is a finite graph whose vertices are connected by $0, 1, 2$ or 3 edges.
- A **Coxeter Graph of a Root System** with respect to Δ has the elements of Δ as vertices and $n(\alpha, \beta)n(\beta, \alpha)$ edges between α and β.

Definition: Irreducible Root Systems

Let $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ and $\Sigma \subset \mathbb{E} \setminus \{0\}$ a root system in $\mathbb{E}$. If $\Sigma_i := \Sigma \cap \mathbb{E}_i$ is a root system on $\mathbb{E}_i$ for $i = 1, 2$, we say that Σ is **reducible**. If not, Σ is **irreducible**.

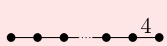
Fact. (1) Any root system can be decomposed into a sum of irreducible root systems.

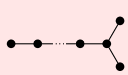
(2) Σ is irreducible if and only if its Coxeter graph is connected.

Theorem III.23

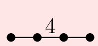
Every connected non-empty Coxeter graph of a root system is isomorphic to one of the following:

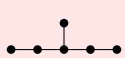
A_n:

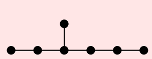
B_n: 

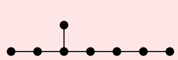
D_n: 

G_2:

F_4: 

E_6: 

E_7: 

E_8: 

Idea of the Proof: Take a Coxeter graph G with vertex set Σ. Define a bilinear form

$(\cdot,\cdot)$ on $\mathbb{R}^{|\Sigma|}$ on the basis $\{e_\alpha\}_{\alpha\in\Sigma}$ by

$$(e_\alpha, e_\beta) = \begin{cases} \cos\left(\frac{\pi}{2}\right) & \alpha \text{ and } \beta \text{ are connected by 0 vertices} \\ \cos\left(\frac{2\pi}{3}\right) & \alpha \text{ and } \beta \text{ are connected by 1 vertex} \\ \cos\left(\frac{3\pi}{4}\right) & \alpha \text{ and } \beta \text{ are connected by 2 vertices} \\ \cos\left(\frac{5\pi}{6}\right) & \alpha \text{ and } \beta \text{ are connected by 3 vertices} \end{cases}$$

If Σ is irreducible, there exists a unique (up to a constant) inner product on $\mathbb{E}$. This will lead to the classification. ■

The Coxeter graphs do not specify everything. We still need the relative length of the roots to the following matrix.

Definition: Cartan Matrix

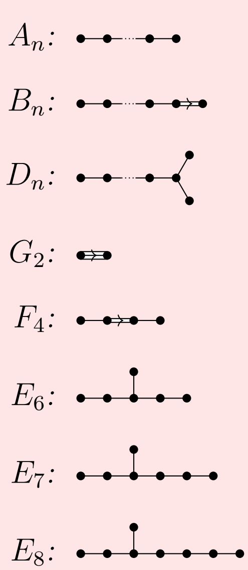
If Σ is a root system, the **Cartan Matrix** of Σ with respect to a basis Δ is the matrix with entries $(n(\beta,\alpha))_{\alpha,\beta\in\Delta}$.

Fact. A reduced root system is defined up to isometry by its Cartan matrix.

We put weights on each vertex α proportional to the length squared $\langle\alpha,\alpha\rangle$. The diagram obtained in this way is called the **Dynkin diagram of** Σ. Two proportional Dynkin diagrams describe the same Cartan matrix.

Theorem III.24

SAME DIAGRAMS AS BEFORE? Each non-empty connected Dynkin diagram is isomorphic to one of the following



Remark. This is a classification of the simple complex Lie algebras as any simple complex Lie algebra $\mathfrak{g}$ determines an irreducible root system, hence a Cartan matrix and a Dynkin diagram. Conversely, any Dynkin diagram defines a Cartan matrix which determines an irreducible root system and therefore a simple complex Lie algebra $\mathfrak{g}$.

We next wish to construct a basis $\Delta(\gamma)$ starting from a regular element γ such that $\Delta(\gamma) = \Delta(\gamma')$ if γ and γ' are in the same Weyl chamber. To do so, we first consider

$$\begin{aligned}\Sigma^+(\gamma) &\coloneqq \{\alpha \in \Sigma : \langle \alpha, \gamma \rangle > 0\} \\ &= \{\text{roots on the same half-space as } \gamma\}, \\ \Sigma^-(\gamma) &\coloneqq -\Sigma^+(\gamma)\end{aligned}$$

and notice that obviously we have

$$\Sigma = \Sigma^+(\gamma) \sqcup \Sigma^-(\gamma).$$

Definition: Indecomposable Roots

$\alpha \in \Sigma^+(\gamma)$ is called **indecomposable** if it cannot be written as a sum of two elements in $\Sigma^+(\gamma)$. We then define

$$\Delta(\gamma) \coloneqq \{\alpha \in \Sigma^+(\gamma) : \alpha \text{ indecomposable}\}.$$

Remark. For a given basis Δ we can write

$$\Sigma = \Sigma^+ \sqcup \Sigma^-$$

where Σ^+ denotes the set of all roots with non-negative coefficients in the basis Δ, and Σ^- the set of all the roots with non-positive coefficients in the basis Δ.

Theorem III.25

If $\gamma \in \mathbb{E}$ is a regular element, then $\Delta(\gamma)$ is a basis of Σ. Moreover, every basis of Σ is of this form.

Proof. We first show that property (2) of a basis is satisfied by showing that every element in $\Sigma^+(\gamma)$ is a linear combination of elements in $\Delta(\gamma)$ with non-negative integers as coefficients:

We write

$$\{\langle \alpha, \gamma \rangle : \alpha \in \Sigma^+(\gamma)\} = \{0 < s_1 < \cdots < s_\ell\}$$

and assume by contradition that there exists $\alpha \in \Sigma^+(\gamma)$ such that α cannot be written as a linear combination of elements in Δ with non-negative integers as coefficients. Notice that if α were indecomposable, then $\alpha \in \Delta(\gamma)$ by definition, which would be a contradiction to the choice of α.

Among these counterexamples we now take one for which $\langle \alpha, \gamma \rangle$ is minimal, say $\langle \alpha, \gamma \rangle = s_k$, $k \in \{1, 2, \dots, \ell\}$. As α is decomposable we can write $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Sigma^+(\gamma)$, hence

$$s_k = \langle \alpha, \gamma \rangle = \underbrace{\langle \beta_1, \gamma \rangle}_{>0} + \underbrace{\langle \beta_2, \gamma \rangle}_{>0},$$

which implies $k \neq 1$ and $\langle \beta_i, \gamma \rangle \leq s_{k-1}$ for $i = 1, 2$. But as any $\beta \in \Sigma^+(\gamma)$ with $\langle \beta, \gamma \rangle \leq s_{k-1} < s_k$ can be written as a linear combination of elements in Δ with non-negative integers as coefficients, the same holds for $\alpha = \beta_1 + \beta_2$, which is a contradiction to our assumption.

We next show that $\Delta(\gamma)$ is linearly independent: We write

$$0 = \sum_{\alpha \in \Delta(\gamma)} \lambda_\alpha \alpha$$

and consider the sets

$$\begin{aligned} \Delta^+ &= \{\alpha \in \Delta(\gamma) : \lambda_\alpha > 0\}, \\ \Delta^- &= \{\beta \in \Delta(\gamma) : \lambda_\beta < 0\}. \end{aligned}$$

Our goal will be to show that $\Delta^+ = \Delta^- =$. We observe that

$$\sum_{\alpha \in \Delta^+} \lambda_\alpha \alpha = - \sum_{\beta \in \Delta^-} \lambda_\beta \beta = \sum_{\beta \in \Delta^-} |\lambda_\beta| \beta$$

and thus

$$\begin{aligned} \left\| \sum_{\alpha \in \Delta^+} \lambda_\alpha \alpha \right\|^2 &= \langle \sum_{\alpha \in \Delta^+} \lambda_\alpha \alpha, \sum_{\beta \in \Delta^-} |\lambda_\beta| \beta \rangle \\ &= \sum_{\alpha \in \Delta^+, \beta \in \Delta^-} \lambda_\alpha |\lambda_\beta| \langle \alpha, \beta \rangle. \end{aligned}$$

Notice that if $\alpha, \beta \in \Delta(\gamma)$ are distinct, then $\langle \alpha, \beta \rangle \leq 0$. Otherwise Lemma III.21 would imply $\beta - \alpha \in \Sigma^+(\gamma)$ or $\alpha - \beta \in \Sigma^+(\gamma)$, hence

$$\beta = (\beta - \alpha) + \alpha \in \Sigma^+ \qquad \text{or} \qquad \alpha = (\alpha - \beta) + \beta \in \Sigma^+,$$

which contradicts that α and β are indecomposable. So we conclude that

$$\left\| \sum_{\alpha \in \Delta^+} \lambda_\alpha \alpha \right\|^2 = \sum_{\alpha \in \Delta^+, \beta \in \Delta^-} \lambda_\alpha |\lambda_\beta| \langle \alpha, \beta \rangle \leq 0,$$

hence $\sum_{\alpha\in\Delta^+} \lambda_\alpha \alpha = 0$, so by definition Δ^+ we get $\Delta^+ =$. Similarly we obtain from

$$\sum_{\beta\in\Delta^-} |\lambda_\beta|\beta = \sum_{\alpha\in\Delta^+} \lambda_\alpha \alpha = 0$$

and the definition of Δ^- that $\Delta^- =$.

We finally consider any other basis $\Delta \subset \Sigma$ of $\mathbb{E}$. It is an easy exercise to show that there is exists $\gamma \in \mathbb{E}$ such that

$$\langle \alpha, \gamma \rangle > 0 \qquad \text{for all} \quad \alpha \in \Delta.$$

Then clearly $\gamma \in \mathbb{E}$ is regular, and moreover

$$\Sigma^+ \subseteq \Sigma^+(\gamma) \qquad \text{and} \quad \Sigma^- \subseteq \Sigma^-(\gamma).$$

From

$$\begin{aligned} \Sigma &= \Sigma^+ \cup \Sigma^- \\ &= \Sigma^+(\gamma) \cup \Sigma^-(\gamma) \end{aligned}$$

it follows that we have $\Sigma^\pm = \Sigma^\pm(\gamma)$. Since $\Delta(\gamma)$ consists of indecomposable elements we have $\Delta(\gamma) \subset \Delta$ and thus $\Delta(\gamma) = \Delta$. ■

Remark. If γ, γ' belong to the same Weyl chamber, then $\Delta(\gamma) = \Delta(\gamma')$ since

$$\operatorname{sign}(\langle \alpha, \gamma \rangle) = \operatorname{sign}(\langle \alpha, \gamma' \rangle) \qquad \forall \alpha \in \Sigma.$$

It follows that there is a one-to-one correspondence between bases and Weyl chambers.

Definition: Weyl Group of Root System

If $\alpha \in \Sigma$ and σ_α denotes the reflection with respect to P_α, then

$$W := \langle \sigma_\alpha : \alpha \in \Sigma \rangle < \mathrm{O}(\mathbb{E})$$

is called the **Weyl Group of** Σ.

Remark. If $w \in W$, then $w(\Sigma) = \Sigma$ and for every $w \in W$ and $\alpha \in \Sigma$ we have

$$w \sigma_\alpha w^{-1} = \sigma_{w(\alpha)}.$$

Thus W permutes the Weyl chambers and in fact, if γ is regular and $C(\gamma)$ denotes the Weyl chamber containing γ, then

$$w(C(\gamma)) = C(w(\gamma))$$

Moreover, if Δ is a basis, then $w(\Delta)$ is also a basis for all $w \in W$. Hence W also permutes the set of basis of Σ. Notice that if C is a Weyl chamber containing γ and γ', then

$$\Sigma^+(\gamma) = \Sigma^+(\gamma') \qquad \text{and} \quad \Delta(\gamma) = \Delta(\gamma').$$

This gives a W-equivariant map between the set of Weyl chambers and the set of basis of Σ.

Theorem III.26

Let Δ be a basis of Σ. Then

(1) $W = \langle \sigma_\alpha : \alpha \in \Delta \rangle$,

(2) W acts simply transitively on the set of bases and

(3) W acts simply transitively on the set of Weyl chambers.

Lemma III.27

Let $\alpha \in \Delta \subset \Sigma$, where Σ is a reduced root system. Then σ_α permutes $\Sigma^+ \backslash \{\alpha\}$.

Proof. Let $\beta \in \Sigma^+ \backslash \{\alpha\}$ and write

$$\beta = \sum_{\delta \in \Delta} c_\delta \delta \qquad \text{with} \quad c_\delta \in \mathbb{Z}_{\geq 0}.$$

As Σ is reduced, β is not a multiple of α and thus $c_\delta \neq 0$ for some $\delta \neq \alpha$. But

$$\sigma_\alpha(\beta) = \beta - \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

and thus the coefficient of δ in $\sigma_\alpha(\beta)$ is the same as the coefficient of δ in β. Thus $\sigma_\alpha(\beta) \in \Sigma^+$ and since also $\sigma_\alpha(\beta) \neq \alpha$ we have

$$\sigma_\alpha(\beta) \in \Sigma^+ \backslash \{\alpha\}.$$

■

Lemma III.28

If $\beta = \sum_{\delta \in \Sigma^+} \delta$ then

$$\sigma_\alpha(\beta) = \beta - 2\alpha \qquad \text{for all} \quad \alpha \in \Delta.$$

Proof. Observe that $\beta = \sum_{\delta \in \Sigma^+ \setminus \{\alpha\}} \delta + \alpha$, hence by the previous Lemma

$$\begin{aligned} \sigma_\alpha(\beta) &= \sigma_\alpha\left(\sum_{\delta \in \Sigma^+ \setminus \{\alpha\}} \delta + \alpha\right) \\ &= \sum_{\delta \in \Sigma^+ \setminus \{\alpha\}} \delta + \underbrace{\sigma_\alpha(\alpha)}_{-\alpha} \\ &= \beta - 2\alpha. \end{aligned}$$

■

Lemma III.29

Take $\alpha_1, \ldots, \alpha_n \in \Delta$ (not necessarily distinct) and write $\sigma_i := \sigma_{\alpha_i}$, $i = 1, ..., n$. Assume $\sigma_1 \cdots \sigma_{n-1}\sigma_n(\alpha_n) \in \Sigma^+$. Then there exists $i \in \{1, 2, \ldots, n-1\}$ such that

$$\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \hat{\sigma}_i \cdots \sigma_{n-1}.$$

Proof. Note that $\sigma_n(\alpha_n) = -\alpha_n$ implies that $\sigma_1 \cdots \sigma_{n-1}(\alpha_n) \in \Sigma^-$. We then distinguish two cases:

1. If $\sigma_{n-1}(\alpha_n) \in \Sigma^-$, then $\alpha_n = \alpha_{n-1}$ because σ_α permutes $\Sigma^+ \setminus \{\alpha\}$ for all $\alpha \in \Delta$. But then $\sigma_n = \sigma_{n-1}$ and hence

$$\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \sigma_{n-2}.$$

2. If $\sigma_{n-1}(\alpha_n) \in \Sigma^+$ then let $1 \leq i \leq n-2$ be the smallest index such that

$$\sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma^+ \qquad \text{and} \qquad \sigma_i\sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma^-.$$

Then with $w := \sigma_{i+1} \cdots \sigma_{n-1} \in W$ we have $w(\alpha_n) = \alpha_i$ and thus

$$\sigma_i = \sigma_{\alpha_i} = \sigma_{w(\alpha_n)} = w\sigma_{\alpha_n}w^{-1} = w\sigma_n w^{-1}.$$

So we get

$$\sigma_{i+1} \cdots \sigma_{n-1}\sigma_n = w\sigma_n = \sigma_i w = \sigma_i\sigma_{i+1} \cdots \sigma_{n-1},$$

and if we multiply this on the left by $\sigma_1 \cdots \sigma_i$ this finally yields

$$\sigma_1 \cdots \sigma_i\sigma_{i+1} \cdots \sigma_n = \sigma_1 \cdots \underbrace{\sigma_i\sigma_i}_{=Id}\sigma_{i+1} \cdots \sigma_{n-1}.$$

■

From this lemma we immediately get the following

Corollary III.30

Let $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ where n is the minimal number of factors in a product decomposition of σ as a product of reflections in Δ. Then

$$\sigma(\alpha_n) \in \Sigma^-.$$

Proof of Theorem III.26. We write $W' = \langle \sigma_\alpha : \alpha \in \Delta \rangle$.

1. First we show that W' acts transitively on the set of Weyl chambers. We need the following

   Claim: Let $\gamma \in \mathbb{E}$ be regular. Then there exists $\sigma \in W'$ such that

   $$\langle \sigma(\gamma), \alpha \rangle > 0 \qquad \forall \alpha \in \Delta.$$

   Assuming the claim to be true, then transitivity follows from the fact that $\sigma(C(\gamma)) = C(\sigma(\gamma))$.

   Proof of Claim: Let $\delta = \sum_{\alpha \in \Sigma^+} \alpha$ and let $\sigma \in W$ such that $\langle \sigma(\gamma), \gamma \rangle$ is maximal. Then for all $\alpha \in \Delta$

   $$\begin{aligned} \langle \sigma(\gamma), \delta \rangle &\geq \langle \sigma_\alpha(\sigma(\gamma)), \delta \rangle \\ &= \langle \sigma(\gamma), \sigma_\alpha(\delta) \rangle \\ &= \langle \sigma(\gamma), \delta - 2\alpha \rangle \\ &= \langle \sigma(\gamma), \delta \rangle - 2\langle \sigma(\gamma), \alpha \rangle. \end{aligned}$$

   Thus $\langle \sigma(\gamma), \alpha \rangle \geq 0$ and hence $\langle \sigma(\gamma), \alpha \rangle > 0$ for all $\alpha \in \Delta$ since $\sigma(\gamma)$ is regular as well.

2. Now we prove that for any $\alpha \in \Sigma$ there exists $\sigma \in W'$ such that $\sigma(\alpha) \in \Delta$.

   Indeed, as W' acts transitively on the set of bases, it is enough to show that any $\alpha \in \Sigma$ belongs to some basis. To do this, let $\gamma \in P_\alpha \setminus \cup_{\beta \in \Sigma \setminus \{\pm\alpha\}} P_\beta$. Pick γ' close to γ such that $\langle \gamma', \alpha \rangle = \varepsilon > 0$ and

   $$|\langle \gamma', \beta \rangle| > \varepsilon \qquad \forall \beta \in \Sigma \setminus \{\pm\alpha\}$$

   Thus α is indecomposable and belongs to $\Sigma^+(\gamma')$ and hence to $\Sigma^+(\gamma)$.

3. We next show that $W' = W$, which amounts to show that for any $\alpha \in \Sigma$ we have $\sigma_\alpha \in W'$. If $\alpha \in \Sigma$, then according to the second step there exists $\sigma \in W'$ such that $\sigma(\alpha) \in \Delta$ and hence

   $$\sigma_\alpha = \sigma^{-1} \sigma_{\sigma(\alpha)} \sigma \in \sigma^{-1} W' \sigma = W'.$$

4. We finally show that W acts freely on the set of basis. Let $\sigma \in W = W'$, $\sigma \neq e$ be such that $\sigma(\Delta) = \Delta$. Write $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ as a minimal product of simple reflections $\alpha_1, \dots, \alpha_n \in \Delta \subset \Sigma^+$ with $n \geq 2$. Then $\sigma(\alpha_n) \in \Sigma^-$ which is a contradiction to $\sigma(\Delta) = \Delta \subset \Sigma^+$. ∎

III.5 Iwasawa Decomposition

Let Σ be a root system associated to an orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ coming from a Riemannian symmetric pair (G, K). Denote $\mathfrak{g}_\alpha$ the root spaces of $\alpha \in \mathfrak{a}^* \backslash \{0\}$ where $\mathfrak{a} \subset \mathfrak{p}$ is the maximal Abelian subalgebra. Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$,

$$\Sigma^+ = \{\alpha \in \Sigma : \alpha(H) > 0 \; \forall H \in \mathfrak{a}^+\}.$$

Set then

$$\mathfrak{n}^+ := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}$ is finite dimensional, $\mathfrak{n}^+$ is nilpotent and $N^+ := \exp(\mathfrak{n}^+)$ is a unipotent subgroup of G.

Theorem III.31: Iwasawa Decomposition

We have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

If $N^+ := \exp(\mathfrak{n}^+)$ and $A = \exp(\mathfrak{a})$, then the map

$$\begin{aligned} K \times A \times N^+ &\to G \\ (k, a, n) &\mapsto kan \end{aligned}$$

is a diffeomorphism.

Sketch of the proof: Let $X \in \sum_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha$. Since $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ we can write

$$X = \underbrace{X + \Theta(X)}_{\in \mathfrak{k}} - \underbrace{\Theta(X)}_{\in \mathfrak{n}^+}$$

showing that

$$\sum_{\alpha \in \Sigma^-} \subset \mathfrak{k} + \mathfrak{n}^+.$$

Moreover, since $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$, we have

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{g}_0 \cap \mathfrak{g} \\ &= \mathfrak{g}_0 \cap \mathfrak{p} \oplus \mathfrak{g}_0 \cap \mathfrak{k} \\ &= (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}. \end{aligned}$$

Thus we finally get

$$\begin{aligned}\mathfrak{g} = \underbrace{\mathfrak{g}_0}_{=(\mathfrak{g}_0\cap\mathfrak{k})\oplus\mathfrak{a}} + \underbrace{\sum_{\alpha\in\Sigma^-}\mathfrak{g}_\alpha}_{\subset\mathfrak{k}+\mathfrak{n}^+} + \underbrace{\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha}_{=\mathfrak{n}^+} \\ \subset \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}^+.\end{aligned}$$

■

Remark. Let $M = G/K$ be a Riemannian symmetric space with base point $o \in M$. Then

$$M = N^+ \cdot A \cdot K \cdot o = N^+ \cdot A \cdot o$$

is a foliation of M by the flats $A \cdot o$.

Example. $(\mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(n,\mathbb{R}))$ with

$$\mathfrak{a}^+ = \{\mathrm{diag}(t_1,\ldots,t_n) : \sum t_j = 0,\ t_1 > \cdots > t_n\}.$$

We saw that there are $n(n-1)$ roots

$$\alpha_{ij}(H) = t_i - t_j \qquad \text{for} \qquad H \in \mathfrak{a}^+$$

with corresponding root spaces $\mathfrak{g}_{ij} = \mathbb{R}E_{ij}$. Then

$$\Sigma^+ = \{\alpha_{ij} : i < j\}$$

and a basis is $\Delta = \{\alpha_{i,i+1} : i = 1,\ldots,n-1\}$. Then $\mathfrak{n}^+ \subset \mathfrak{sl}(n,\mathbb{R})$ corresponds to the strictly upper-triangular matrices and $N^+ \subset \mathrm{SL}(n,\mathbb{R})$ corresponds to the upper triangular matrices with ones on the diagonal. Finally

$$A = \left\{\mathrm{diag}(\lambda_1,\ldots,\lambda_n) : \prod \lambda_i = 1\right\}.$$

so that

$$\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)\cdot A\cdot N^+$$

which is essentially Gram-Schmidt.

Appendix A

Lie Groups

Definition: Lie Group

A *Lie group* is a smooth manifold that is also a group and such that the group operations are smooth. The *dimension* of a Lie group is the dimension of the underlying manifold.

Example. (1) $(\mathbb{R}^n, +)$;

(2) The *general linear group*

$$\mathrm{GL}(n,\mathbb{R}) := \{g \in \mathbf{R}^n : \det g \neq 0\}.$$

It is disconnected, and the two connected components are characterized by the sign of the determinant. In particular $\mathrm{GL}(1,\mathbb{R}) = (\mathbb{R}^*, \cdot)$.

(3) The *special linear group*

$$\mathrm{SL}(n,\mathbb{R} := \{g \in \mathrm{GL}(n,\mathbb{R}) : \det g = 1\}$$

is connected and not compact since it contains the matrix $g_s := \begin{pmatrix} s & 0 & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$.

(4) Let $B_p\, \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the symmetric bilinear form of signature (p,q), where $q = n-p$ and let Q_p be the associated quadratic form. Let

$$\mathrm{O}(p,q) := \{g \in \mathrm{GL}(n,\mathbb{R}) : Q_p(gv) = Q_p(v) \text{ for all } v \in \mathrm{R}^n\}.$$

We claim that $\mathrm{O}(p,q)$ is compact if and only if $p=0$. We illustrate the argument in the case in which $p=1$ and the general case will be clear. If $(e_1,\dots,e_n)$ is the standard basis of $\mathbb{R}^n$, let us consider the new basis $(e_1', e_2', e_3, \dots, e_n)$, where

$e_1' = e_2 - e_1$ and $e_2' = e_2 + e_1$. With respect to this new basis the quadratic form Q_p becomes $Q_p'(v) = -v_1^2 + \sum_{j=2}^n v_j^2$. It is clear that the matrix g_s in (3) is in $\mathrm{O}(1, n-1)$ which is hence not compact.

If on the other hand $p = 0$, the bilinear form B_0 is nothing but the usual inner product in $\mathbb{R}^n$ and the group $\mathrm{O}(0, n)$ is called the *(real) orthogonal group* and is usually denoted by $\mathrm{O}(n, \mathbb{R})$. If $g \in \mathrm{O}(n, \mathbb{R})$, we can write $g = ((c_1)\dots(c_n))$, where for $i \leq j \leq n$ the c_j are $n \times 1$ column vector such that $c_j = ge_j$. Then $(c_1, \dots, c_n)$ is an orthonormal basis in $\mathbb{R}^n$, so that $\|c_j\| = 1$ for all $1 \leq j \leq n$. Thus $|g_{ij}| \leq 1$ for all $1 \leq i, j \leq n$ and hence $\mathrm{O}(n, \mathbb{R})$ is compact.

(5) The unitary group

$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n, \mathbb{C}) :\ g^* g = I\},$$

where $g^* = {}^t\overline{g}$, is compact while the complex orthogonal group

$$\mathrm{O}(n, \mathbb{C}) := \{g \in \mathrm{GL}(n, \mathbb{C}) :\ {}^t g\, g = I\}$$

is not.

(6) If $B \colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ is the standard non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$

$$B(x, y) := \sum_{1 \leq p \leq n} (x_p \overline{y_{n+p}} + \overline{x_{n+p}} y_p)$$

the *symplectic group* is

$$\begin{aligned} \mathrm{Sp}(2n, \mathbb{C}) :=& \{g \in \mathrm{SL}(2n, \mathbb{C}) :\ B(x, y) = B(gx, gy) \text{ for all } x, y \in \mathbb{C}^{2n}\} \\ =& \{g \in \mathrm{SL}(2n, \mathbb{C}) :\ {}^t g\, F\, g = F\}\,, \end{aligned}$$

where $F = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

(7) The product and the semidirect product of Lie groups is a Lie group.

(8) The *n-dimensional torus* $\mathbb{T}^n$ is an Abelian Lie group, where

$$\mathbb{T} := \{z \in \mathbb{C} :\ |z| = 1\}\,,$$

and can be identified with

$$\mathrm{SO}(2, \mathbb{R}) := (O)(2, \mathbb{R}) \cap \mathrm{SL}(2, \mathbb{R}) = \{g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\}$$

via the isomorphism $e^{\imath\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(9) All countable discrete groups are (zero-dimensional) Lie groups.

(10) Every subgroup of a Lie group is a Lie group.

(11) Every Lie group is locally isomorphic to a linear Lie group, that is to a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

(12) The universal covering $\widetilde{\mathrm{SL}(n, \mathbb{R})}$ of $\mathrm{SL}(n, \mathbb{R})$ is a Lie group but not linear, that is there are no faithful representations into $\mathrm{GL}(n, \mathbb{R})$. In fact, as manifolds, $\mathrm{SL}(n, \mathbb{R})$ is diffeomorphic to $\mathrm{SO}(n, \mathbb{R}) \times \mathbb{E}^k$ for some $k \in \mathbb{N}$. Then

$$\pi_1(\mathrm{SL}(n, \mathbb{R}) = \pi_1(\mathrm{SO}(n, \mathbb{R}) = \begin{cases} \mathbb{Z}_2 & n \geq 3 \\ \mathbb{Z} & n = 2\,. \end{cases}$$

Thus $\widetilde{\mathrm{SL}(n, \mathbb{R})}$ is the two-sheeted cover of $\mathrm{SL}(n, \mathbb{R})$ if $n \geq 3$ and is the infinite-sheeted cover of $\mathrm{SL}(2, \mathbb{R})$ if $n = 2$.

Definition: Lie Algebra

A *Lie algebra* $\mathfrak{g}$ is a finite dimensional vector space endowed with a bilinear antisymmetric operation $[\ ,\ ] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *bracket*, that for all $X, Y, Z \in \mathfrak{g}$ satisfies the *Jacobi identity*

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]\,.$$

We say that $\mathfrak{g}$ is *Abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example. (1) Any associative algebra is a Lie algebra with the bracket

$$[a, b] = ab - ba\,.$$

(2) The space of $n \times n$ matrices is a Lie algebra with the bracket induced by the matrix multiplication.

(3) The vector space $\mathrm{Vect}(M)$ of smooth vector fields on a manifold M is a Lie algebra in which the operation is the bracket of vector fields.

(4) The vector space $\mathbb{R}^3$ is a Lie algebra with the cross product.

Let G be a Lie group and let L_g be the left translation on G. We say that a vector field $X \in \mathrm{Vect}(G)$ is *left invariant* if $(L_g)_* X = X$. It is easy to see that the space $\mathrm{Vect}(G)^G$ of left invariant vector fields is closed under the bracket operation and is hence a Lie subalgebra of $\mathrm{Vect}(G)$ thus a Lie algebra as well. It is easy to see, via the evaluation map, that $\mathrm{Vect}(G)^G$ can be identified with the tangent space to G at the identity and is hence finite dimensional.

Definition: Lie Algebra of a Lie group

The *Lie algebra of a Lie group* G is the vector space $\mathrm{Vect}(G)^G$ of left invariant vector fields on the group. The Lie algebra of a Lie group G is denoted by $\mathfrak{g}$ or $\mathrm{Lie}(G)$.

Definition: Lie group and Lie agebra homomorphisms

(1) A *Lie group homomorphism* is a group homomorphism that is smooth.

(2) A *Lie algebra homomorphism* is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.

Recall that any measurable homomorphism of a locally compact topological group is continuous and, if the group is actually a Lie group, it is smooth.

In the case of a linear Lie group the identification with the tangent space at the identity has also the very nice consequence that the bracket of vector fields can be computed as the bracket of matrices in the following sense:

Proposition A.1:

The Lie algebra $\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$ *is isomorphic as Lie algebra to* $\mathbb{R}^{n\times n}$ *with the usual bracket of matrices.*

The following result gives the relationship between Lie group homomorphisms and Lie algebra homomorphisms. Recall that if G, H are topological groups, a local homomorphism is a map $\varphi\colon U \to H$, where $U \subset G$ is an open neighborhood of $e \in G$, such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in U$ such that $xy \in U$.

Proposition A.2:

(1) *If $\varphi\colon G \to H$ is a Lie group homomorphism, then $d_e\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

(2) *If G, H are Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, and $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then there exists a local homomorphism $\varphi\colon B \to H$ such that $d_e\varphi = \pi$.*

(3) *If the homomorphism in (2) is an isomorphism, the local homomorphism is a local isomorphism.*

(4) *If two simply connected Lie groups have isomorphism Lie algebras, then they are isomorphic.*

The hypothesis that the Lie groups in (4) are simply connected is essential as the following example shows:

Example. Let $\varphi\colon \mathbb{R} \to S^1$ bt eh homomorphism $t \mapsto e^{\imath t}$. Then $d_0\varphi\colon \mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(S^1)$ is a Lie algebra isomorphism and so is also $(d_0\varphi)^{-1}\colon \mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$. If $(d_0\varphi)^{-1}$ were the differential of a homomorphism $\psi\colon S^1 \to \mathbb{R}$, $\psi(S^1)$ would be a one-dimensional compact subgroup of $\mathbb{R}$. But this is impossible since the only compact subgroup of $\mathbb{R}$ is trivial. Hence $(d_0\varphi)^{-1}$ is the differential of a local isomorphism.

Definition: One-parameter Subgroup

A *one-parameter subgroup* of a Lie groups is a Lie group homomorphism $\varphi\colon \mathbb{R} \to G$, that is a smooth curve that is also a homomorphism.

The hypothesis of simple connectivity plays in our favor in this case. In fact, given a Lie group G and $X \in \mathfrak{g}$ we can define a Lie algebra homomorphism $\mathrm{Lie}(\mathbb{R}) \to \mathfrak{g}$ defined as $t \mapsto tX$. By Proposition **??** there exists a local homomorphism that, since $\mathbb{R}$ is simply connected, extends to a global homomorphism. We denote by φ_X the one-parameter subgroup such that

$$\varphi_X(0) = e \quad \text{and} \quad \dot{\varphi}_X(0) = X\,.$$

The link between lines in the Lie algebra and one-paramenter subgroups in the Lie group is given by the exponential map.

Definition: Exponential Map

The *exponential map* of the Lie algebra $\mathfrak{g}$ (or equivalently, of the Lie group G) is defined as

$$\begin{aligned} \exp_{\mathfrak{g}} \colon \mathfrak{g} &\longrightarrow G \\ X &\mapsto \varphi_X(1)\,. \end{aligned}$$

When there is no risk of confusion we might drop the subscript $\mathfrak{g}$.

Example. (1) If $G = S^1$, then $\exp\colon \mathrm{Lie}(S^1) \to S^1$ is, the usual exponential map $\exp(t) = e^{\imath t}$.

(2) In the case of the general linear group, the exponential map is, once again, just the usual exponential of a matrix

$$\exp(X) = e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!}\,.$$

Proposition A.3: Properties of the exponential map

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$.

1. $\exp(t_1 + t_2)X = \exp(t_1 X)\exp(t_2 X)$ *for all* $t_1, t_2 \in \mathbb{R}$.
2. $\exp(tX)^{-1} = \exp(-tX)$ *for all* $t \in \mathbb{R}$.
3. $\exp : \mathfrak{g} \to G$ *is a smooth map, and a local diffeomorphism from a neighborhood of* $0 \in \mathfrak{g}$ *onto a neighborhood of* $e \in G$*. In fact,* $d_0 \exp = Id$.

If $h\colon G \to G$ is any homomorphism, then, chasing the definitions, one can check the naturality of the Lie group exponential map. In other words, the following diagram commutes

$$\begin{CD} \mathfrak{g} @>{\mathrm{d}_e h}>> \mathfrak{g} \\ @V{\exp}VV @VV{\exp}V \\ G @>>{h}> G\,, \end{CD}$$

that is, for all $g \in G$ and $X \in \mathfrak{g}$,

$$\exp(\mathrm{d}_e h X) = h(\exp X)\,. \tag{A.1}$$

A particularly important homomorphism is the conjugation $c_g\colon G \to G$, defined by $c_g(h) := ghg^{-1}$. This is a Lie group isomorphism whose differential

$$\mathrm{Ad_G}(g) := \mathrm{d}_e c_g\colon \mathfrak{g} \to \mathfrak{g}$$

at the identity $e \in G$ is a Lie algebra automorphism, that is

$$\mathrm{Ad_G}(g)([X,Y]) = [\mathrm{Ad_G}(g)(X), \mathrm{Ad_G}(g)(Y)]$$

for all $X, Y \in \mathfrak{g}$ and $g \in G$. In particular (A.1) applied to $h = c_g$ yields

$$\exp(\mathrm{Ad_G}(g)X) = g \exp X g^{-1}\,. \tag{A.2}$$

The map

$$\mathrm{Ad_G}\colon G \to \mathrm{GL}(\mathfrak{g})$$

is a group homomorphism called the *Adjoint representation of* G. Taking this one step further, its derivative at the identity

$$\mathrm{ad}_\mathfrak{g} := \mathrm{d}_e \mathrm{Ad_G}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is the *adjoint representation of* $\mathfrak{g}$. This is a Lie algebra homomorphism, that is

$$\mathrm{ad}_\mathfrak{g}([X,Y]) = [\mathrm{ad}_\mathfrak{g}(X), \mathrm{ad}_\mathfrak{g}(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Neither $\mathrm{Ad_G}$ nor $\mathrm{ad}_\mathfrak{g}$ are necessarily faithful. Their respective kernels are the center $Z(G)$ of G and the center $\mathfrak{z}(\mathfrak{g})$, which satisfy the relation $\mathrm{Lie}(Z(G) = \mathfrak{z}(\mathfrak{g})$.

Appendix B

Preliminaries

B.1 Topological Preliminaries

Definition

Let X by a a topological space and (Y, d) a metric space. A family $\mathcal{F} \subset C(X, Y)$ of functions if *equicontinuous* is for every $x \in$X and every $\epsilon > 0$, there exists an open $U_x \subset X$ such that for all $f \in \mathcal{F}$ and all $x' \in U_x$, $d(f(x), f(x')) < \epsilon$.

Theorem B.1: Ascoli–Arzelà

Let X be a topological space and (Y, d) a metric space. Give $C(X, Y)$ the compact-open topology and let $\mathcal{F} \subset C(X, Y)$. Then:

(1) $\mathcal{F}$ is equicontinuous and the set $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ has compact closure for all $a \in X$ then $\mathcal{F}$ is relatively compact.

(2) The converse holds is X is locally compact and Hausdorff.

Definition

A *(topological) fiber bundle* $\mathcal{B}$ consists of

(1) a topological space B (*bundle space*),

(2) a topological space X (*base space*),

(3) a continuous map $P\colon B \twoheadrightarrow X$ (*projection*),

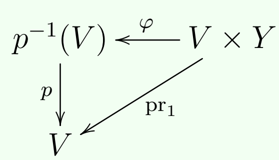
(4) a topological space Y (*fiber*) such that for every $x \in X$ the *fiber* $p^{-1}(x)$ must be homeomorphic to Y.

Moreover a fiber bundle is *locally trivial*, that is

(5) for every $x \in X$, there exists a neighborhood V of x and a homeomorphism

$$\varphi\colon V \times Y \times p^{-1}(V)$$

such that the diagram



commutes, that is $p\varphi(x', y) = x'$ for all $x' \in V$ and $y \in Y$.

Finally a *cross-section* of $\mathcal{B}$ is a a continuous map $\sigma\colon X \to B$ that is a right inverse to p, that is such that $p \circ \sigma(x) = x$ for all $x \in X$.

We consider now the case in which B is a group, so that X ia a G-homogeneous space $X = G/H$ and $Y = H = \mathrm{Stab}_G(p)$. This is what is called a *principal bundle*. In this case having a *local* cross-section implies (5). In fact we can define $\varphi\colon V \times H \to p^{-1}(V)$ as

$$\varphi(x, h) \coloneqq \sigma(x)h$$

and it is easy to verify that $p \circ \varphi(x, h) = x$. For the inverse φ^{-1} we have the formula

$$\varphi^{-1}(p^{-1}(x)) = (x, x^{-1}\sigma(p(x)))\,,$$

where we need to verify that $x^{-1}\sigma(p(x)) \in H$. In fact, by definition of σ,

$$p(x^{-1}\sigma(p(x))) = p(x^{-1})p(\sigma(p(x))) = p(x^{-1})p(x) = p(x^{-1}x) = p(e) \in H\,.$$

B.2 Differential Geometrical Preliminaries (added as we move along, no logical order...)

Lemma B.2:

Let M be a Riemannian manifold and $p_0 \in M$. Then there exists a ball $B_r(p_0)$ that is a normal neighborhood of each of its points, with the following property. Let $p, q \in B_r(p_0)$ and let $\gamma\colon [0,1] \to M$ the unique geodesic in $B_r(p_0)$ joining $p = \gamma(0)$ and $q = \gamma(1)$ and let $\mathrm{L}(\gamma)$ be its length. Then:

(1) For any (p', v) near $(p, \dot\gamma(0))$ with $v \in T_{p'}M$, $\|v\| = 1$, there exists a geodesic γ' in $B_r(p_0)$ of the same length as $\mathrm{L}(\gamma)$ starting at p' and with v as tangent vector in p'.

(2) The data (p', v) (and hence the geodesic γ') depends smoothly on $(p, \dot\gamma(0))$.

B.2.1 Completeness

Theorem B.3: Hopf–Rinow

Let (M, g) be a Riemannian manifold. The following assertions are equivalent:

(1) M is geodesically compete, that is, all geodesics are defined over $\mathbb{R}$ or, equivalently, Exp_p is defined on T_pM for all $p \in M$;

(2) There exists $p \in M$ such that Exp_p is defined on T_pM;

(3) (M, d) is complete as a metric space.

(4) The closed bounded subsets of M are compact.

Moreover, each of these assertions implies the existence of a minimizing geodesic between any two given points.

B.2.2 Connections

Definition

A C^∞ or *affine connection* ∇ on a differentiable manifold (M, g) is a map $\nabla\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$, $(X, Y) \to \nabla_X Y$ with the properties that for every $f, f' \in C^\infty(M)$ and every $X, X', Y, Y' \in \mathrm{Vect}(M)$,

(1) $\nabla_{fX+f'X'}Y = f(\nabla_X Y) + f'(\nabla_{X'} Y)$;

(2) $\nabla_X(fY + f'Y') = f\nabla_X Y + f'\nabla_X Y' + (Xf)Y + (Xf')Y'$.

Remark. (1) A C^∞ connection ∇ is $\mathbb{R}$-linear in both variables, but it is $C^\infty(M)$-linear only in the first variable and not in the second one.

(2) Another difference between the role that the two variables X, Y play, is reflected in the fact that the value at the point $p \in M$ of the vector field $\nabla_X Y$ depends only on the value X_p of the vector field X at p, but not on the vector field X. (The same is not true of the dependence on Y.)

A connection allows to differentiate vector fields defined along curves. If $\gamma\colon \mathbb{R} \to M$ is a smooth curve, we call $\nabla_{\dot\gamma} X$ the *covariant derivative* of X along γ.

Definition

We say that a vector field X along a curve γ is *parallel* if $\nabla_{\dot\gamma} X = 0$.

While "constant" vector fields, that is vector fields $Y \in \mathrm{Vect}(M)$ such that at every point $p \in M$ $(\nabla_X Y)_p = 0$ for every $X \in \mathrm{Vect}(M)$ rarely exists, it follows from the existence and uniqueness of the solutions of differential equations always exist:

Proposition B.4

Let M be a differentiable manifold. Given a curve γ and a vector $v \in T_{\gamma(0)}M$, there exists a unique vector field X^v parallel along γ such that $X_{\gamma(0)} = v$.

The *parallel transport* along γ from $\gamma(0)$ to $\gamma(t)$ is defined as the linear isomorphism $T_{\gamma(0)}M \to T_{\gamma(t)}M$ given by $v \mapsto (X^v)_{\gamma(t)}$. This gives an identification of the tangent spaces at $\gamma(0)$ and at $\gamma(t)$. Geodesics in a differentiable manifolds are defined as differentiable curves $\gamma\colon I \to M$ such that $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all $t \in I \subset \mathbb{R}$.

One could add more conditions to the ones defining an affine connection. An affine connection satisfying also condition (3) below is called a symmetric connection.

Definition

An affine connection that in addition satisfies

(3) it is *symmetric*, namely $[X,Y] = \nabla_X Y - \nabla_Y X$, and

(4) $Xg(Y,Y') = g(\nabla_X, Y, Y') + g(Y, \nabla_X Y')$

is called *Riemannian connection.*

Theorem B.5: Fundamental Theorem in Riemannian Geometry

Given a Riemannian manifold (M,g), there exists a unique Riemannian connection, called the Levi-Civita connection.

The following lemma is not at all surprising:

Lemma B.6:

Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve, Y a parallel vector field along γ and $f \in \mathrm{Iso}(M)$. Then $f_* Y$ is a parallel vector field along $f \circ \gamma$.

Proof. Let us consider the map $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by $(X,Y) \mapsto f_*^{-1}(\nabla_{f_* X} f_* Y) =: D_X Y$. If we show that $D_X Y$ satisfies (1) through (4) of Definition **??**, then by Theorem **??**, $\nabla_X Y = f_*^{-1}(\nabla_{f_* X} f_* Y)$, so that $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$. If $X = \dot\gamma$, then $f_*(\nabla_{\dot\gamma} Y) = \nabla_{f_*\dot\gamma} f_* Y = \nabla_{(f\circ\gamma)^{\cdot}} f_* Y$, so that $\nabla_{(f\circ\gamma)^{\cdot}} f_* Y = 0$ if $\nabla_{\dot\gamma} Y = 0$.

Properties (1) and (2) are obvious. To see (3), recall that $[f_* X, f_* Y] = f_*[X,Y]$, so that

$$\nabla_{f_* X} f_* Y - \nabla_{f_* Y} f_* X = [f_* X, f_* Y] = f_*[X,Y] = f_*(\nabla_X Y - \nabla_Y X)\,.$$

It follows that

$$[X,Y] = \nabla_X Y - \nabla_Y X = D_X Y - D_Y X\,,$$

so that (3) is verified. Then (4) follows from the following chain of equalities:

$$\begin{aligned}
& g(D_X Y, Y') + g(Y, D_X Y') \\
=& g(f_*^{-1}(\nabla_{f_* X} f_* Y), Y') + g(Y, f_*^{-1}(\nabla_{f_* X} f_* Y')) \\
=& g(\nabla_{f_* X} f_* Y, f_* Y') + g(f_* Y, \nabla_{f_* X} f_* Y') \\
=& (f_* X) g(f_* Y, f_* Y') \\
=& f_* X) f_*(g(Y, Y')) \\
=& X g(Y, Y')\,.
\end{aligned}$$

■

In fact, this is only a particular case of the fact that if $f\colon M \to M$ is a diffeomorphism and $\nabla\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is an affine connection, then $D\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by $D_X Y \coloneqq f_*^{-1}(\nabla_{f_* X} f_* Y)$ is also an affine connection. In particular, if M is a Lie group and $f \coloneqq L_g$ is the left translation via $g \in G$, then a connection that satisfies

$$\nabla_X Y \coloneqq (L_g)_*^{-1}(\nabla_{(L_g)_* X}(L_g)_* Y) \tag{B.1}$$

is called *left invariant*.

Here is a result about the differential of the exponential map associated to an affine connection. Recall that an affine connection is *analytic* if the map $p \mapsto (\nabla_X Y)_p$ is analytic for any two analytic vector fields $X, Y \in \mathrm{Vect}(M)$.

If $X \in T_p M$, for $p \in M$, we denote by X^* the vector field defined on a normal neighborhood around $p \in M$, obtained by parallel translation of X along a geodesic joining two points.

Recall that there exist neighborhoods $N_q(M)$ of $q \in M$ and $V_q M$ of $0 \in T_q M$ such that $\mathrm{Exp}_q\colon V_q \to N_q M$ is a diffeomorphism. The differential at $X \in V_q M$ will hence be: $d_X(\mathrm{Exp}_\nabla)_q\colon T_X(V_q M) \to T_{(\mathrm{Exp}_\nabla)_q(X)} N_q(M)$ or else, by identifying $T_X(V_q M)$ with $T_q M$,

$$d_X(\mathrm{Exp}_\nabla)_q\colon T_q M \to T_{(\mathrm{Exp}_\nabla)_q(X)}(M)\,.$$

An analogous relation holds for tX, provided that t is small enough that $tX \in V_q M$.

If $X \in T_q M$, for $q \in M$, we denote by X^* the vector field defined on a normal neighborhood around $q \in M$, obtained by parallel translation of X along a geodesic joining two points.

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Theorem B.7: Helgason

Theorem I.6.5, Helgason Let M be an analytic manifold with an analytic connection. Let $q \in M$ and $X \in T_q M$. Then there exists $\epsilon > 0$ such that for $Y \in T_q M$,

$$(d_{tX}(\mathrm{Exp}_\nabla))(Y) = \left(\sum_{n=0}^{\infty} \frac{\theta(-tX^*)^n}{(n+1)!}(Y^*) \right)_{(\mathrm{Exp}_\nabla)(tX)}$$

for $|t| < \epsilon$, where $\theta(X) \coloneqq [X, Y]$.

B.2.3 Curvature

We know that if X, Y are vector fields, $[X, Y]$ measures the extent to which X and Y do not commute. We can also define a quantity that measures the extent to which

$\nabla_X Y$ and $\nabla_Y X$ do not commute, by adding also a term that depends on $[X, Y]$ and "makes things better".

Definition

Let M be a manifold with an affine connection. The *curvature* of M is a multilinear mapping (when $\mathrm{Vect}(M)$ is considered as a $C^\infty(M)$-module) $R\colon \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by

$$R(X,Y)Z \coloneqq \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(Z)\,.$$

To every $X, Y \in \mathrm{Vect}(M)$, it associates the *curvature operator*

$$R(X,Y)\colon\ \mathrm{Vect}(M) \to \mathrm{Vect}(M)$$

It follows from the presence of the term $\nabla_{[X,Y]}$ that at each point $p \in M$ the vector $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p and not on their values in a neighborhood of p. Thus R defines a linear transformation $R(X_p, Y_p)\colon T_pM \to T_pM$ and in fact $R\colon T_pM \times T_pM \to \mathrm{Lin}(T_pM)$ is a map that to two vectors at the point p, associates a linear operator from T_pM into itself.

If M is a Riemannian manifold, then the Riemannian metric allows us to see the curvature as a (4,0)-tensor, by setting $R(X,Y,Z,T) = g(R(X,Y)Z,T)$.

The Riemann curvature tensor has the following symmetries:

(R_1) $R(X,Y,Z,T) = -R(Y,X,Z,T) = R(Z,T,X,Y)$;

(R_2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (First Bianchi Identity)

It is not difficult to see that the curvature tensor and the curvature operator completely determine each other.

Given a Riemannian manifold, there are other notions of curvature. The *sectional curvature* $K(P)$ of a 2-plane P in T_pM is defined as follows. If $\{u, v\}$ is an orthonormal basis of P (orthonormal with respect to the Riemannian metric g) then

$$K(P) \coloneqq -R(u,v,u,v)\,. \tag{B.2}$$

The sectional curvature coincides with the usual notion of Gaussian curvature on a surface. Namely, if P is a tangent 2-plane in T_pM and Σ is a portion of surface in M tangent to P at p, then the sectional curvature of P is exactly the Gaussian curvature of Σ at p.

Moreover the symmetry properties of the Riemann curvature tensor imply that it can be completely determined by knowing the sectional curvature on all sections of T_pM.

B.2.4 Totally Geodesic Submanifolds

Definition

Let M be a Riemannian manifold and $N \subset M$ a connected submanifold. Let $p \in N$ The submanifold N is *geodesic at* p if given any tangent vector $v \in T_pN$, the M-geodesic $\gamma_v\colon -(\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ is contained in N.
The submanifold N is *totally geodesic* if it is geodesic at every point $p \in N$.

It is not difficult to show that then the M-geodesic $\gamma \subset N$ is also an N-geodesic and that any N-geodesic is also an M-geodesic. As a consequence, if M is complete, then N is complete.

Totally geodesic submanifolds in Riemannian manifolds are not frequent. If $M = \mathbb{R}^n$, then linear subspaces and their translates are totally geodesic. If $M = S^n \subset \mathbb{R}^{n+1}$, then the intersection of S^n with linear subspaces are totally geodesic. It was proven by Cartan, that if a Riemannian manifold M has the property that for every $p \in M$ and for every two-dimensional plane $P \subset T_pM$, there exists a totally geodesic submanifold tangent to P, then M has constant curvature.

Theorem B.8:

Let M be a Riemannian manifold and N a connected complete submanifold. Then N is totally geodesic if and only if the M-parallel transport along curves in N sends tangent vectors to N to tangent vectors to N.

One direction of the above theorem is obvious if we replace "curve" with "geodesic".

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